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A Causal Net approach to Relativistic Quantum Mechanics and Gravitation

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Abstract

In this paper we discuss a causal network model to describe quantum mechanics and gravitation, where space–time is built solely from point events and connecting probabilities. A causal net provides a representation which combines the quantum complementary features of both particles and waves as each vertex on the causal net represents a possible point event or particle observation. Simultaneity on the causal net is defined by hyperplanes of equivalent proper time. The causal net model when constructed in Minkowksi space–time is shown to lead to a formulation for relativistic quantum mechanics including the Dirac equation. In a curved Riemann space–time the causal paths of the causal net are geodesics and in the local Lorentz frame the net probabilities and the Dirac formalism are preserved. The variation of space–time density of events in the causal paths modifies the metric and provides a space–time curvature leading to the Hilbert action associated with general relativity. Consideration of the Schwarzchild metric shows that in classical limit the perturbation in the density of possible events is proportional to the Newtonian gravitational potential.

1 Introduction

Causality, or the concept of "cause and effect", has long been viewed by philosophers as a fundamental and often an *a priori* principle in our understanding and interpretation of nature. Although Newton's laws were clearly written in causal terms, the indeterminacy of quantum mechanics lead to confusion of the role of causality in describing quantum systems.

In the causal net approach to relativistic quantum mechanics [1] space–time is built on the primitives of a set of "possible point events" and causal relations. A causal net provides a representation which combines the quantum complementary features of both particles and waves and hence describes wave–particle duality. The causal net is constructed in the local Lorentz frame and simultaneity on the causal net is defined for events lying on hyperplanes of equivalent proper time. The causal net space–time discretisation method exactly derives the Dirac equation [2] and provides all the common fermion features, such as spin, negative energy states, action of a potential and summation of paths [3]. The most basic causal net describes a plane wave solution with the space axis aligned along the direction of momentum. The causal net is similar to the Lorentz invariant *Aether* of causally connected events proposed by Dirac in 1951 [4] and is consistent with Dirac's gauge which describes the existence and motion of a classical electron with no self interaction [5].

Einstein himself speculated on the possibility of a causal theory underlying general relativity and in 1916 he wrote "Causal set theory arises by combining discreteness and causality to create a substance that can be the basis of a theory of quantum gravity. Spacetime is thereby replaced by a vast assembly of discrete "elements" organised by means of "relations" between them...None of the continuum attributes of spacetime, neither metric, topology nor differentiable structure are retained, but emerge it is hoped as approximate concepts at macroscopic scales."

For curved space–time the causal paths of the causal net are geodesics. In the general relativity formalism the curved space is mapped by the causal net, as originally envisaged by Einstein, as a grid of geodesics [6,7]. The Dirac equation in the local Lorentz frame is unchanged in these geodesic coordinates and the geometry and probabilities linking events are preserved. The curvature of space–time is produced by varying the space–time density of events leading to a modified coordinate system for observers. Overlapping events from different causal nets, can increase the density of events and provide a causal net analogue to gravitation. In the non-relativistic limit the additional density of events is proportional to the Newtonian gravitational potential.

2 The causal net approach to relativistic quantum mechanics

2.1 A causal net in space–time

As a preliminary I will present an overview of the causal net approach to relativistic quantum mechanics detailed in [1]. To construct the causal net for a particle motion in space–time, consider a 1 dimensional space aligned with the direction of particle motion, and embedded in 3+1 dimensional space. In this 1 dimensional space the simplest causal net that satisfies the definition of simultaneity is a 1+1 dimensional "diamond" lattice with causal links connecting the lattice points as in Figure 3. Each causal connection is defined by a connecting arrow giving a definite lineal order and an associated probability. Each vertex on the causal net represents a possible event — meaning a possible observation of the particle — and has two incoming and two outgoing causal connections so that each event has two effective possible common causes. Starting at a vertex and following an outgoing arrow at random at each subsequent vertex describes a "causal chain" as a series of possible events.

Measurement or observation at a vertex or a region of the net provides, through Bayesian statistics, a re-evaluation of these probabilities after a measurement. A causal net of possible events thus constitutes a simple Bayesian network. This is illustrated in Figure 1 where an event at $\mathcal{A}$ is more likely to have been caused by an event at $\mathcal{B}$ than $\mathcal{C}$ and $\mathcal{D}$ is an impossibility due to zero connectivity between the paths. Bayes theorem provides a way of translating this common sense concept into a formal probabilistic context since $P(\mathcal{A} \mid \mathcal{B}) > P(\mathcal{A} \mid \mathcal{C}) > P(\mathcal{A} \mid \mathcal{D})$. On measurement of an event this Bayesian re-evaluation and reassessment of probabilities is analogous to the well known "collapse of the wavefunction" in other interpretations of quantum mechanics.

First, consider the simple case of representing a particle randomly diffusing on the 1+1 dimension causal net shown in Figure 1. The net is then made up of elementary triangles labelled with $(\Delta x, c\Delta t, c\Delta\tau)$ as shown in Figure 2. To guarantee invariance of causality on the net we impose c as the speed of light [8]. Since, from geometry, $\Delta x/c\Delta t = \sin\theta \leq 1$, and then identify Δt as relativistic time intervals in an observer frame and $\Delta\tau$ as the particle proper time interval. The proper time interval is the actual time experienced by a particle or a local Lorentz observer moving between the two events and simultaneity on the causal net is then defined for events lying on hyperplanes of equivalent proper time . The net geometry guarantees the invariant space–time interval

$$(c\Delta\tau)^2 = (c\Delta t)^2 - (\Delta x)^2. \tag{1}$$

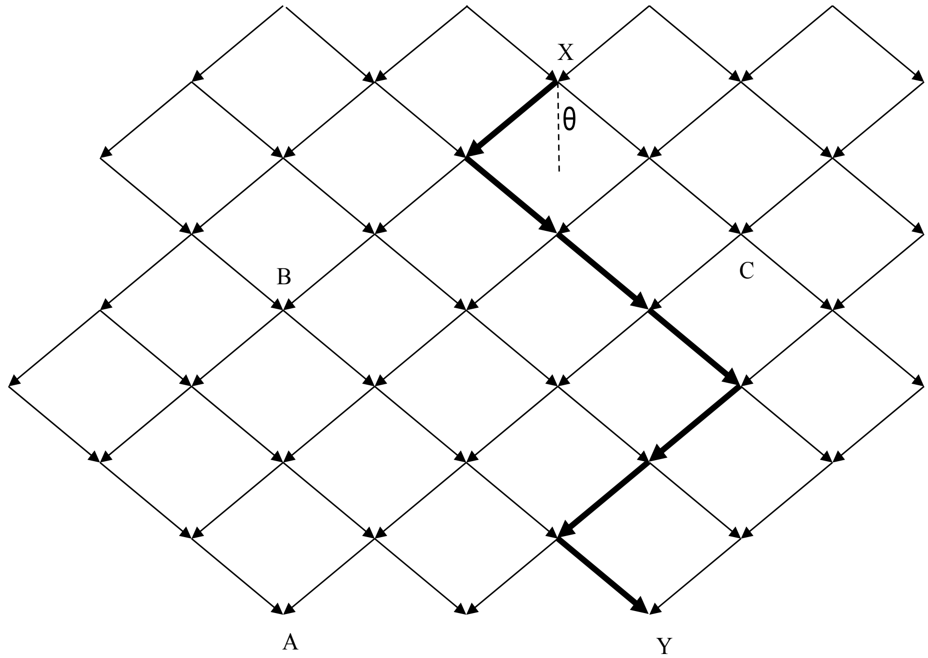

Figure 1: Causal net showing causal chain from X to Y. The net angle θ is shown.

On the discrete causal net we define the observed velocity in terms of finite differences. The definition adopted is $v = \Delta x/\Delta t$ which we equate to the expectation of the velocity on the causal net. The two time intervals are then related by $\Delta\tau = \Delta t/\gamma$ where $\gamma = 1/\sqrt{1 - v^2/c^2}$ is the Lorentz factor specifying the net angle θ

$$\cos\theta = 1/\gamma \qquad \sin\theta = v/c. \tag{2}$$

Associated with each event the 4-vector velocity $v_\mu = \gamma(c, \vec{v})$ provides a commuting invariant relation

$$v^\mu v_\mu = c^2. \tag{3}$$

This invariant relation is the starting point for developing the mathematics of a causally connected series of events in space–time and was first proposed in 1951 by Dirac [4] as defining a Lorentz invariant Aether. In Dirac's formulation at each point in space-time the velocity of the Aether is subject to quantum uncertainty and is potentially multi–valued. The velocity is badly defined but may be described by a probability distribution and a pure isotropic empty vaccuum state is not measurable. We shall see that Dirac's Aether is consistent with the causality described by our Lorentz invariant network of possible events in space–time.

Clearly Eq. (1) and thus the net can be scaled by a factor. If we identify this with the particle rest mass m then from Eq. (1) we then have the relativistic dispersion relation

$$E^2 = p^2c^2 + m^2c^4, \tag{4}$$

where E is the particle energy $E = \gamma mc^2$ and $p = \gamma mv$ the momentum. We can further rearrange to derive a second useful invariant relation

$$-mc^2\Delta\tau = p\Delta x - E\Delta t, \tag{5}$$

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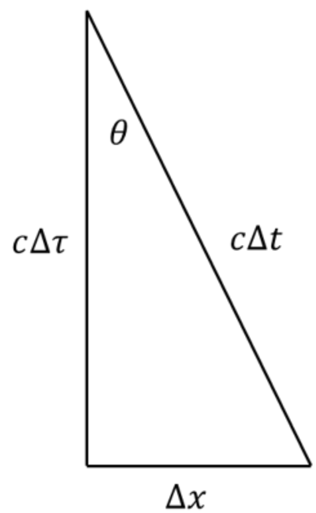

Figure 2: The elementary space-time "triangle" for the causal net built in the local Lorentz frame .

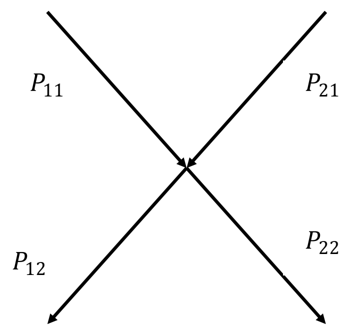

Figure 3: A vertex (1,2) on the causal net with associated probabilities.

and a third, the Lagrangian for a free particle

$$L = -mc^2/\gamma = pv - E = pv - H, \tag{6}$$

where H is the Hamiltonian.

If we identify the causal net as representing an electron with charge e and velocity v moving in an electromagnetic potential (A_0, A_1) with

$$eA_0 = E, \tag{7}$$

and

$$eA_1 = pc. \tag{8}$$

The net automatically describes the movement of an electron in an electromagnetic potential with the dispersion relation

$$-m^2c^4 = e^2{A_1}^2 - e^2{A_0}^2. \tag{9}$$

The above 1+1 dimension case is for the field component A_1 aligned along the causal net x space axis. In 3+1 dimensions this corresponds to Dirac's gauge condition [5]

$$A^\mu A_\mu = k^2 = (mc^2/e)^2, \tag{10}$$

where the electron velocity at different points in space–time is directly linked to the local electromagnetic potential. In Dirac's classical theory there is no electron without momentum and without a field and vice versa. The gauge connects directly with Maxwell's equations for the electromagnetic field.

2.2 The uncertainty principle and action

From our definition of simultaneity and the geometry of the net the $\sum p\Delta x$ is the same on the lattice for all paths between two events which implies that $p\Delta x$ is a constant for a valid causal net. Identifying the net constant with Planck's constant h provides the de Broglie relation [9] with $\Delta x = \lambda/2$

$$\lambda p = h, \tag{11}$$

and a Heisenberg like relation [10,11]

$$\Delta p \Delta x \sim h/2. \tag{12}$$

The discrete nature of the net automatically entails a de Broglie relation and an uncertainty principle. The spatial separation of events for a given proper time is equal to the de Broglie wavelength.

Starting with the relation Eq. (5) we can consider the causal path sum of this relation over n events between two non adjacent distant event which leads to several action principles. The Maupertuis action S_M, that does not explicitly involve the time taken between events in space, can be written as a function of the number of steps between events n and h

$$S_M = \sum_i^n p\Delta x = nh. \tag{13}$$

We can also derive a more general action principle S_τ based on proper time and related to the Lagrangian

$$S_\tau = -\sum_i^n mc^2\Delta\tau = -mc^2 n\Delta\tau = -\sum_i^n L\Delta t, \tag{14}$$

which can approximated for large n with an integral to acquire the well known result

$$S = -\int L\mathrm{dt} = -\int mc^2 \mathrm{d}\tau. \tag{15}$$

2.3 The Dirac equation

The indeterminism on the causal net is governed by Eq. (1). Thus a particle in its own local Lorentz frame in a proper time interval $\Delta\tau$ moving at a speed $|v|$ can move to a position $\pm\Delta x$ over time Δt in an inertial observer frame. This produces a random trajectory in space–time as in Figure 1. Consider an individual vertex on the net and label the incoming probabilities on row 1 P_{11} and P_{21} and outgoing probabilities on row 2 P_{12} and P_{22} as shown in Figure 3. Probability is conserved at the vertex and the total probability at a vertex is given by $P_T = P_{11} + P_{21}$. If the average velocity measured on the lattice is uniform then $P_{11} = P_{22}$ and $P_{12} = P_{21}$. If we consider normalised branching probabilities at the vertex defined as $\hat{P}_{11} + \hat{P}_{21} = 1$ then since expected velocity at the vertex is defined to be v we have

$$E[v] = \gamma\frac{\Delta x}{\Delta t}[\hat{P}_{11} - \hat{P}_{21}] = v. \tag{16}$$

The branching probabilities are then given by

$$\hat{P}_{11} = \frac{E + mc^2}{2E} \qquad \hat{P}_{21} = \frac{E - mc^2}{2E}. \tag{17}$$

From this we can see that in the low velocity limit $|v| \to 0$ then $\hat{P}_{11} \to 1$ and $\hat{P}_{21} \to 0$ and in the high velocity limit $|v| \to c$ then $\hat{P}_{11} \to \hat{P}_{21} \to 1/2$. At higher velocities the causal path followed becomes more random and the trajectory exhibits "Zitterbewegung". The expected velocity v is equivalent in both local Lorentz and observer frames if time and space are represented as orthogonal basis vectors. The probabilities can also be written simply in terms of net angles

$$\hat{P}_{11} = \cos^2(\theta/2) \qquad \hat{P}_{21} = \sin^2(\theta/2). \tag{18}$$

Each real probability can be formed by combining complex probability amplitudes $P_{ij} = \phi_{ij}.\phi_{ij}^*$ with

$$\phi_{ij} = \sqrt{P_{ij}} e^{-imc^2\tau/\hbar} = \sqrt{P_{ij}} e^{i(px-Et)/\hbar}, \tag{19}$$

which depend on the proper time τ at the net vertices and x and t are defined at the discrete net vertices. The probability amplitudes at each vertex on the net can be expressed in terms of a unique transfer matrix M

$$\Phi = \begin{pmatrix} \phi_{22} \\ \phi_{12} \end{pmatrix} = \begin{pmatrix} \phi_{11} \\ \phi_{21} \end{pmatrix} = M \begin{pmatrix} \phi_{11} \\ \phi_{21} \end{pmatrix}, \tag{20}$$

defined as

$$M = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = \begin{pmatrix} 1/\gamma & v/c \\ v/c & -1/\gamma \end{pmatrix} = \frac{1}{E} \begin{pmatrix} mc^2 & pc \\ pc & -mc^2 \end{pmatrix} = \frac{H_D}{E}. \tag{21}$$

Here we recognise H_D as the Dirac Hamiltonian for a free particle [4, 12] with defined momentum p. To connect with the complete quantum mechanics we note that Eq. (21) can be put in the conventional form [13] by assuming that space–time is locally differentiable at the vertex, allowing us to use the usual momentum operator $\hat{p}$ to replace the momentum eigenvalues p. We can write

$$\begin{pmatrix} mc^2 & c\hat{p} \\ c\hat{p} & -mc^2 \end{pmatrix} \Psi = E\Psi = i\hbar \frac{\partial \Psi}{\partial t}, \tag{22}$$

where we have replaced the probability amplitudes Φ with the familiar 2 component Dirac spinor Ψ for the free particle [12,13]

To extend to the general 3+1 dimension case we must consider transformations of the causal net that leave it invariant under spatial direction of velocity $\vec{v}$. Using polar coordinates then for momentum $\vec{p} = |\vec{p}|(sin\vartheta cos\varphi, sin\vartheta sin\varphi, cos\vartheta)$ the wavefunction components become dependent on the coordinates (ϑ, φ) so $\sqrt{P_{ij}}$ becomes $\sqrt{P_{ij}}\chi(\vartheta, \varphi)$ where $\chi(\vartheta, \varphi)$ is a multiplicative function. Following Dirac's convention [2] we can replace the 1 dimension momentum operator $\hat{p}$ with the 3 dimensional momentum operator $(\vec{\sigma}.\vec{p})$, formed from Pauli matrices σ_k $(k = 1, 2, 3)$. By definition this momentum operator provides the relation $(\vec{\sigma}.\vec{p})\chi_\pm = |\vec{p}|\chi_\pm$ with two eigenvectors

$$\chi_+ = (\cos\vartheta/2, e^{i\varphi}\sin\vartheta/2) \qquad \chi_- = (-e^{-i\varphi}\sin\vartheta/2, \cos\vartheta/2). \tag{23}$$

The general solutions for the wavefunction then become four 4-component orthogonal vectors corresponding to up and down spin $S = \pm 1/2$ with positive and negative energies $\epsilon = \pm 1$. Omitting the phase factors and normalisation constant these are

$$\Psi_{\substack{\epsilon=+1 \\ S=+1/2}} = \begin{pmatrix} \sqrt{P_{11}}\chi_+ \\ \sqrt{P_{21}}\chi_+ \end{pmatrix} \qquad \Psi_{\substack{\epsilon=+1 \\ S=-1/2}} = \begin{pmatrix} \sqrt{P_{11}}\chi_- \\ -\sqrt{P_{21}}\chi_- \end{pmatrix}, \tag{24}$$

and

$$\Psi_{\substack{\epsilon=-1 \\ S=+1/2}} = \begin{pmatrix} -\sqrt{P_{21}}\chi_+ \\ \sqrt{P_{11}}\chi_+ \end{pmatrix} \qquad \Psi_{\substack{\epsilon=-1 \\ S=-1/2}} = \begin{pmatrix} \sqrt{P_{21}}\chi_- \\ \sqrt{P_{11}}\chi_- \end{pmatrix}. \tag{25}$$

These are the common solutions to the conventional 3+1 dimension Dirac equation [12]

$$\begin{pmatrix} mc^2 & c(\vec{\sigma}.\vec{p}) \\ c(\vec{\sigma}.\vec{p}) & -mc^2 \end{pmatrix} \Psi = E\Psi = i\hbar \frac{\partial \Psi}{\partial t}. \tag{26}$$

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2.4 Inertial mass and correspondence principle

The causal net and the separation of events in space and time leads naturally to the concept of mass as being the density of possible events in space–time. Since we have fixed the net constant as Planck's constant h, the absolute size or scale of the causal net in time and space can thus be evaluated as

$$\Delta\tau = \frac{h}{2mc^2(\gamma^2-1)} \qquad \Delta x = \frac{h}{2mc\sqrt{\gamma^2-1}}, \tag{27}$$

with the scaling relation

$$\Delta\tau = \frac{2mc}{h}(\Delta x)^2. \tag{28}$$

Thus the net decreases in size with increasing velocity or energy as the de Broglie wavelength decreases. Also the scale of the net is also inversely proportional to the mass. Particles of higher mass will have more finely resolved causal nets. Importantly, objects that are large relative to the net size will transition to a more classical behaviour for observers and this resolution effect in the causal net approach provides a *correspondence principle* between the classical and quantum domains.

From Eq. (27) for the distance between events on the causal net we can see that for equivalent velocity, and hence similar net triangles, the net size scales inversely with mass. This is shown schematically in Figure 4. For two masses m_1 and m_2 we have

$$\frac{m_1}{m_2} = \frac{\Delta x_2}{\Delta x_1} = \frac{\Delta t_2}{\Delta t_1}. \tag{29}$$

The number density of events on the net with distance or time is thus proportional to the particle mass. If σ_1 and σ_2 are the density of events in space or time their ratio is given by

$$\frac{\sigma_1}{\sigma_2} = \frac{m_1}{m_2}. \tag{30}$$

Thus, in the causal net model the concept of inertial mass is thus directly equivalent to the density of possible causal events.

We can see that the concept of mass on a causal net and simultaneity provides a link to Mach's principle, whereby it was postulated distant events or far away nebula can influence physics on a local scale. From Eq. (27) for causal paths with relativistic velocities, the size of the causal net is reduced and the net angle $\theta = \arcsin(\frac{v}{c})$ increases towards $\frac{\pi}{2}$. For a free particle, causal events over a finite proper time τ but that are at very large cosmic distances (since $x = c\tau\tan\theta$) can have a causal influence on local events whilst remaining simultaneous with closer events. However, due to the smaller net size such spatially distant causal nets must traverse many intermediate events on their passage and their probability of influencing local events is much lower than for nearby events as in Figure 5.

3 Causal nets and gravitation

3.1 Geodesic causal paths

We shall assume that space-time is Reimann as assumed in the general relativity of Einstein. In the language of general relativity the space–time interval is given by the metric $g_{\mu\nu}$ so $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. Previously, we have considered the special case of the Minkowski metric $\eta_{\mu\nu}$ for flat space–time but general relativity considers Riemann spaces that have quadratic metric equations and are

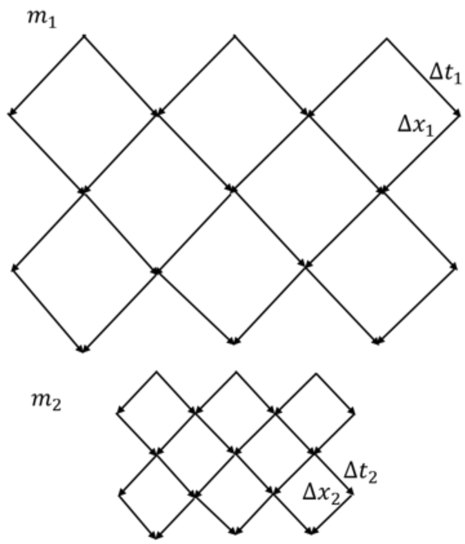

Figure 4: Causal nets for same velocity and hence net angle for mass m_1 and m_2 where $m_2 > m_1$.

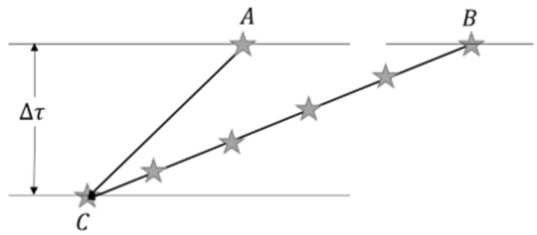

Figure 5: An event at B may be much further away in space and time than event A from event C but may be simultaneous in proper time and have an influence on C despite traversing many events in the causal path.

characterised as locally flat. The causal net is built in the local Lorentz frame with geodesics connecting possible events. A hypothetical observer moving along the path between two events experiences motion along a straight line with constant velocity. We can construct a causal net for Minkowski space–time and then embed the causal net into curved space–time with new coordinates. The coordinates of events in any curved space–time will thus change for any observer not in the geodesic local Lorentz frame. The curved space–time will then be criss-crossed by a grid of geodesics linking events.

The causal net for each velocity is built using a so-called Riemann normal coordinate system [7] with the following characteristics close to an event at $\mathcal{P}$

$$g_{\alpha\beta}(\mathcal{P}) = \eta_{\alpha\beta} \tag{31}$$

$$g_{\alpha\beta,\mu}(\mathcal{P}) = 0. \tag{32}$$

Thus close to any point or event the Riemann space–time is effectively Minkowski and described by the Minkowski metric. Further relations can be derived for the case of normal coordinates close to $\mathcal{P}$ to give the Riemann tensor as the second order derivative of the metric. The causal net of events and the probabilities connecting events, and hence the spinor components will be preserved in moving from flat to curved space–time since there exists local Lorentz invariance in the vicinity of every possible event in Reimann space. This means that at each event space–time is Euclidean $g_{\alpha\beta}(\mathcal{P}) = \eta_{\alpha\beta}$ and the angle θ of the causal net is preserved for a given geodesic velocity. The path visibly taken between events in the observer frame is irrelevant for a geodesic local Lorentz observer that travels along a straight path with constant velocity and arrives at each event with the unchanged net angle θ. The transfer matrix M for the net Eq. (21) and the subsequent results extending to 3+1 dimensions remain unchanged. At each event the branching probabilities are preserved for the plane wave state and hence the Dirac equation is maintained. This representation of the Dirac equation in the local Lorentz frame is much simpler than the representation in the inertial frame of an arbitrary observer who must impose a suitable basis and coordinate system for measurement. In the latter case, mathematical techniques called verbein fields are often introduced

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to transform to the new frame and coordinate system and this leads to complex forms of the Dirac equation. However, the key result here is that in the geodesic frame, represented by the causal net, the Dirac equation is unchanged and retains its simple, elegant form.

As for the Minkowski metric, for simultaneity we must consider hyperplanes between events that have constant proper time, that is the time experienced travelling along the geodesic between events is the same. A path of extremal τ is straight with constant velocity in every local Lorentz frame and is a geodesic of space–time. In general, for curved space–time we can conventionally write the proper time interval between events $\mathcal{A}$ and $\mathcal{B}$ as

$$\tau = \int_{\mathcal{B}}^{\mathcal{A}} \mathrm{d}\tau = -\int_{\mathcal{B}}^{\mathcal{A}} (g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu})^{1/2}. \tag{33}$$

By standard deformation and maximising or extremal lapse of proper time [6,7] we have the geodesic equation in general coordinates at an event $\mathcal{P}$ at x^{α} as

$$\frac{\partial^2 x^{\alpha}}{\partial\tau^2} = -\Gamma^{\alpha}{}_{\mu\nu}\frac{\partial x^{\mu}}{\partial\tau}\frac{\partial x^{\nu}}{\partial\tau} = 0. \tag{34}$$

At the event $\mathcal{P}$ we have $-\Gamma^{\alpha}{}_{\mu\nu} = 0$ leading to a secondary relation

$$g_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial\tau}\frac{\partial x^{\beta}}{\partial\tau} = -c^2. \tag{35}$$

From this equation we can see how the metric $g_{\alpha\beta}$ influences the coordinate system x^{α} for the causal net. To illustrate, we return to the 1+1 dimension causal net case and assume that observers can chose a orthogonal local Lorentz coordinate system to diagonalise the metric. This diagonalisation separates the time and space components of the metric. For flat space–time we have

$$-(c\Delta\tau)^2 = \eta_{00}(c\Delta t)^2 + \eta_{11}(\Delta x)^2, \tag{36}$$

and for curved space–time in 1+1 dimension with time and space coordinates $\hat{t}$ and $\hat{x}$,

$$-(c\Delta\tau)^2 = g_{00}(c\Delta\hat{t})^2 + g_{11}(\Delta\hat{x})^2. \tag{37}$$

If simultaneity between events on the net is determined, as before, by their equivalent separation in proper time $\Delta\tau$, these equations can be rearranged to give

$$\gamma^2(c^2 + \frac{\eta_{11}(\Delta x)^2}{\eta_{00}(\Delta t)^2}) = \gamma^2(c^2 + \frac{g_{11}(\Delta\hat{x})^2}{g_{00}(\Delta\hat{t})^2}) = v^{\mu}v_{\mu} = c^2. \tag{38}$$

Provided the proper time interval is preserved $\Delta\tau$ and the net velocity is equivalent to the ratio

$$(\frac{\eta_{11}(\Delta x)}{\eta_{00}(\Delta t)}) = (\frac{g_{11}(\Delta\hat{x})}{g_{00}(\Delta\hat{t})}) = -v, \tag{39}$$

the causal net paths remain geodesics of constant velocity. Any geodesic trajectory through curved space–time can be mapped piecewise to a Minkowski causal net built in the local Lorentz frame with net triangles as in Figure 6.

This generalises to the previous geodesic relation Eq. (33) for the observer frame in 3+1 dimensions if time and space can be separated. The use of orthogonal local Lorentz coordinate systems allows time and space to be separated in obeying simultaneity. This results in the causal net obeying the relativistic energy dispersion relations in all measurable reference frames. Experimentally, measurements of the universality of the relativistic energy dispersion relations, such as those of electrons in the Crab Nebula [14] have placed tight constraints on any deviations from the well known energy dispersion relations, as Eq. (4), and their invariant quantities.

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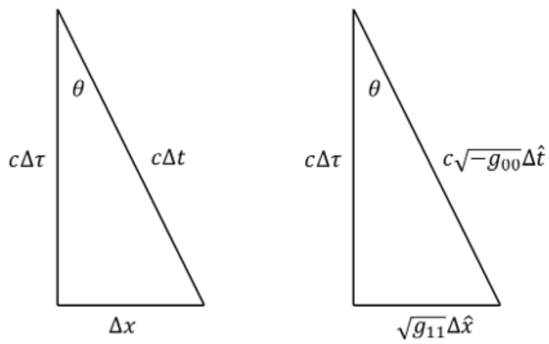

Figure 6: The causal net triangles are equivalent in the local Lorentz frame (left) and the observer frame (right) when scaled by the space–time metric. Curved space–time for geodesics can be mapped to a Minkowski causal net.

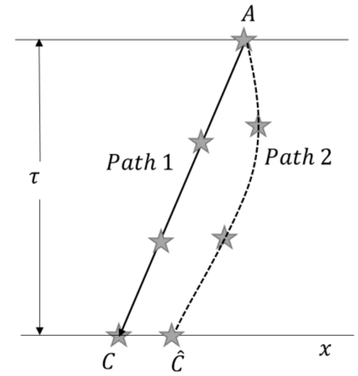

Figure 7: In curved space–time the density of events varies and this leads to an effective metric if simultaneity in proper time is preserved for different causal paths. The number of events in each path (Path 1 and Path 2) is equal for a particular particle mass but the space–time separation between them varies.

3.2 The metric and density of events

The above consideration shows that if a causal net is constructed using geodesics to separate neighbouring events with space–time separations Δx^α and if a metric is applied to curve space then new coordinates $\Delta\hat{x}^\alpha$ must be applied to each event on the net. This effectively "bends" the net in space–time for observers. For the 1+1 dimension case, the distance between events changes such that

$$\Delta\hat{t} = \frac{\Delta t}{\sqrt{-g_{00}}} \qquad \Delta\hat{x} = \frac{\Delta x}{\sqrt{g_{11}}}, \tag{40}$$

and the $\Delta\hat{t}$ and $\Delta\hat{x}$ no longer have a Pythagorean relationship in constructing the causal net but require a metric scaling factor as in Figure 6.

The metric is provided by the density of events along each causal path. Consider the path between two events of total proper time τ for a particle of mass m_1 which comprises n_1 events in the local Lorentz frame. We can write in 1+1 dimensions for a causal net

$$-(\tau c)^2 = -(n_1 c\Delta\tau_1)^2 = \eta_{00}(n_1 c\Delta t_1)^2 + \eta_{11}(n_1\Delta x_1)^2. \tag{41}$$

The absolute number of events n_1 from Eq. (27) can be written as

$$n_1 = \frac{\tau}{\Delta\tau_1} = \frac{2m_1c^2(\gamma^2-1)\tau}{h}. \tag{42}$$

For Minkowski space–time events are distributed homogeneously but the more general case might be where the density of events varies differently in time σ_t and space σ_x. The path can now be written in the new coordinates $\hat{t}$ and $\hat{x}$ as the ratio of density of events between the curved and flat space–time

$$-(\tau c)^2 = -(n_1 c\Delta\tau_1)^2 = \eta_{00}(n_1\frac{\hat{\sigma}_{t1}}{\sigma_1}c\Delta\hat{t}_1)^2 + \eta_{11}(n_1\frac{\hat{\sigma}_{x1}}{\sigma_1}\Delta\hat{x}_1)^2. \tag{43}$$

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The metric then becomes $g_{00_1} = \eta_{00}(\frac{\hat{\sigma}_{t1}}{\sigma_1})^2$ and $g_{11_1} = \eta_{11}(\frac{\hat{\sigma}_{x1}}{\sigma_1})^2$ for 1+1 dimensions. The number of events n_1 is the same over the proper time interval τ but the distance in space and time between them scales with the density of events (see Fig. 7) such that

$$\frac{\sqrt{-\hat{g}_{00_1}}}{\sqrt{-\eta_{00_1}}} = \frac{\Delta t_1}{\Delta \hat{t}_1} = \frac{\hat{\sigma}_{t1}}{\sigma_1} \qquad \frac{\sqrt{\hat{g}_{11_1}}}{\sqrt{\eta_{11_1}}} = \frac{\Delta x_1}{\Delta \hat{x}_1} = \frac{\hat{\sigma}_{x1}}{\sigma_1}. \tag{44}$$

The above results are similar to the well known continuous time definition for the metric based on differentials

$$g_{rs} = \delta_{mn}\frac{\partial x^m}{\partial \hat{x}^r}\frac{\partial x^n}{\partial \hat{x}^s}, \tag{45}$$

but the causal net result assumes a choice of inertial frame coordinates that diagonalises the metric and is based on finite differences.

This methodology extends to 3+1 dimensions and we can write the proper time between events for the curved space–time as

$$-(\tau c)^2 = -(n_1 c\Delta\tau)^2 = \hat{g}_{00_1}(c\Delta\hat{t}_1)^2 + \sum \hat{g}_{ii_1}(\Delta\hat{x}_{ii_1})^2, \tag{46}$$

where

$$\frac{\sqrt{-\hat{g}_{00_1}}}{\sqrt{-\eta_{00_1}}} = \frac{\Delta t_1}{\Delta \hat{t}_1} = \frac{\hat{\sigma}_{t1}}{\sigma_1} \qquad \frac{\sqrt{\hat{g}_{ii_1}}}{\sqrt{\eta_{ii_1}}} = \frac{\Delta x_{i_1}}{\Delta \hat{x}_{i_1}} = \frac{\hat{\sigma}_{i_1}}{\sigma_{i_1}}. \tag{47}$$

The 4-volume elements $\Delta x^4 = \Delta t\Delta x\Delta y\Delta z$ in the two frames are related as

$$\frac{\sqrt{-\hat{g}_1}}{\sqrt{-\eta_1}} = \frac{\Delta x_1^4}{\Delta \hat{x}_1^4}, \tag{48}$$

where g is the trace of the metric. This leads to the common connection between the proper volume in the local Lorentz frame and the observer frame coordinates $\mathrm{d}^4x = \sqrt{-g}\mathrm{d}^4\hat{x}$. The total density of events for each metric multiplied by the 4-volume element is itself constant and since the number of events is proportional to a mass energy this is a restatement of the conservation of energy density.

In most physical situations the metric changes in space and time and this is equivalent to the density of events varying. We have assumed the density of events is constant for all the n_1 events in the path for simplicity of presentation. The changing metric must however be accommodated with a sum or approximated with integrals as Eq. (33).

3.3 The Einstein–Hilbert action

We can see from our simple 1+1 dimension model that the change in density of events modifies the action along the path (Fig. 7). For Minkowski space–time the action is

$$S_1 = \sum_i^{n_1} -m_1c^2\Delta\tau_1 = -n_1m_1c^2\Delta\tau_1 = -m_1c^2\tau. \tag{49}$$

The additional action δS due to a change in the density of events $\delta\sigma$ in the path is can be approximated as a volume integral over an appropriate energy density ρc^2

$$\delta S = -m(\frac{\delta\sigma}{\sigma})c^2\tau \approx -\int \rho c^2\mathrm{d}^3x d\tau. \tag{50}$$

This provides a link to the gravitational action

$$\delta S = -\int \rho c^2 d^3x d\tau = \int T d^4x = -\int \frac{R}{K}d^4x. \tag{51}$$

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Here we have used the trace of the Einstein field equation

$$G = KT = -R, \tag{52}$$

and its relation to the Reimann curvature R and the constant K. The Einstein–Hilbert action [6,7] can be then written in the coordinate system $\hat{x}$ as

$$S_g = \int_{\Omega} T\sqrt{-g}\mathrm{d}^4\hat{x} = -\frac{1}{K}\int_{\Omega} R\sqrt{-g}\mathrm{d}^4\hat{x}. \tag{53}$$

Variation of the Einstein–Hilbert action recovers the complete Einstein field equations in a standard way [6,7]. This demonstrates that the causal net model based on possible events is compatible with Einstein's general theory of relativity.

3.4 The Schwarzchild metric and Newtonian gravity

The spherically symmetric Schwarzchild metric in general relativity is an important metric that can be used to represent the gravitational field of planets, stars and black holes. In spherical coordinates at a distance of r it is conventionally given as

$$-c^2 d\tau^2 = -(1-\frac{R_S}{r})c^2dt^2 + (1-\frac{R_S}{r})^{-1}dr^2 + r^2 d\Omega^2 \tag{54}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric on a unit two sphere and $R_S = 2GM/rc^2$ the Schwarzchild radius with metric mass M [6,7]. To reconstruct for the a causal net in analogy with Eq. (43) we can discretise and relabel the coordinates as observer coordinates.

$$-c^2\Delta\tau^2 = -Ac^2\Delta\hat{t}^2 + A^{-1}\Delta\hat{r}^2 + \hat{r}^2\Delta\hat{\Omega}^2. \tag{55}$$

with $A = \frac{R_S}{\hat{r}}$. Considering a purely radial trajectory for $\Delta\hat{\Omega} = 0$ then metric components can be written

$$\hat{g}_{00} = -A \qquad \hat{g}_{11} = A^{-1}. \tag{56}$$

The coordinates can be mapped to a Minkowski causal net in the local Lorentz frame as

$$\Delta t = A^{1/2}\Delta\hat{t} \qquad \Delta r = A^{-1/2}\Delta\hat{r}. \tag{57}$$

In the limit of large $\hat{r}$ the Schwarzchild solution approaches the Minkwoski causal net. Moving towards the Schwarzchild radius $A \to 0$ so $\Delta\hat{t} \to \infty$ and $\Delta\hat{r} \to 0$ and the observed time to reach the radius from the outside becomes infinity as casual events become closer together. These results are consistent with theories of black holes[7].

Since in the causal net model the metric is related to the density of events from Eq. (47) the relative density of causal events in the time and radial directions can be written as

$$\frac{\hat{\sigma}_t^2}{\sigma^2} = A \qquad \frac{\hat{\sigma}_r^2}{\sigma^2} = A^{-1}. \tag{58}$$

In the non relativistic limit we can expand for a perturbation in the density of events so

$$\frac{\delta\hat{\sigma}_t}{\sigma} = -\frac{\delta\hat{\sigma}_r}{\sigma} \approx -\frac{GM}{\hat{r}c^2} = -\frac{\Phi}{c^2}, \tag{59}$$

where Φ is the Newtonian gravitational potential. Thus approximately, the extra density of possible causal events, above the Minkowski case, is proportional to the classical Newtonian gravitational potential. Using the above action definition in Eq. (49) in the non relativistic limit reconciles with the classical Lagrangian for a particle in a gravitational field

$$L = T - V = \frac{1}{2}mv^2 - m\Phi, \tag{60}$$

ignoring the rest mass term.

4 Discussion

By considering simple casual connections between elementary possible events, based on Reichenbach's principle of common cause, we have constructed a causal model where the Dirac equation and the fermion particles it describes are seemingly "emergent" properties. In the model, fermion particles can arguably be considered as quasiparticles of the causal network composed of possible and actual events analogous to holes and phonons in solid state physics.

The simplest causal network describes exactly the Dirac equation and provides quantum mechanical statistics and major quantum phenomena (diffraction, wave-particle duality, uncertainty ...). In an inertial frame an observer will view an ordered series of events in space–time as an entity behaving as either a wave or a particle, depending on how the experimental measuring setup is conceived, and thus exhibits wave-particle duality exactly as proposed by de Broglie. The quantum measurement "problem" is reduced to simple Bayesian statistics and there is no wavefunction "collapse". The quantum to classical transition becomes a straightforward feature of the resolution of the net which changes with mass and velocity.

Geometrical quantities of the causal net correspond to measurable physical qualities: mass (scaling factor or density of events), momentum and energy (net angle and geometry) and potentials and forces (change of net angle). The global gauge symmetry of the net provides quantum phase and the other degenerate solutions arising from the symmetries of the net are equivalent to the Dirac spin and negative energy states. The discretisation of the net infers similarities to the Heisenberg uncertainty principle and a "stack" of nets provides for different momentum states, consistent with the Feynman path integral approach, quantum phenomenon such as diffraction [1]. The causal net parameters and their physical analogies are outlined in Table 1.

The events in space–time are similar to the Aether described by Dirac in 1951 [4] where the distribution of the 4-velocity is probabilistic and the magnitude of the 4-velocity is relativistically invariant. Although Dirac proposed his Aether, he did not apparently pursue his theory to realise, that a causally connected series of events in his Aether, would lead to the Dirac equation as we demonstrate here. Additionally, the causal net is consistent with the Dirac gauge proposed by Dirac in his theory of the electron [5] whereby classical electrons "appear" and the electron velocity is the local velocity of the Aether. This leads to the interesting concept that perhaps perturbation techniques are not generally required in the description of fermions since the theory is arguably fully renormalisable without infinities, and the physical values such as charge are simply those that are measured.

Mass in the causal net model is provided by the net scaling or density of events in space–time. The density of events has a similar relation to the spatial net spacing or wavelength as a traditional refractive index $\frac{n_1}{n_2} = \frac{m_1}{m_2} = \frac{\lambda_2}{\lambda_1}$. Mach's principle proposed the influence of distant stellar masses on the movement and inertia of local particles. The causal linkages for relativistic particles on the causal net allow such distant events to be simultaneous, as defined by proper time, with more local events. The impact of these distant events, due to their lower probability of local causality, is in most cases small but there nonetheless an influence on local events. The density of events in

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Table 1: Physical analogies of the causal net

Causal net parameters	Physical parameter and concepts
scaling factor	mass
net angle	velocity
net triangle geometry (increased hypotenuse)	energy
net triangle geometry (increased opposite side length)	momentum
change in net geometry	force or changing potential
sum along path	action
global gauge invariance	phase
square root of probabilities (odd and even solutions)	spin
square root of dispersion relation (odd and even solutions)	positive and negative energy states
net discretisation	uncertainty principle
net resolution	correspondence principle
long range causal linkages	Mach's principle
stack of nets	different momentum states, Feynman path integral
density of events in causal path	curved space–time, gravitation

space–time also enters into the metric for the causal net, even for the Minkowski flat metric, which provides an equivalence between inertial and gravitational mass.

For curved space–time the paths of the causal net are geodesics and the Dirac equation in the local Lorentz frame is unchanged in these geodesic coordinates. In the vicinity of an event space–time is locally flat and the causal net angle and probabilities linking events are preserved. The curvature of space–time in the causal net model is produced by variation of the density of possible events in the causal path. If relativistic simultaneity is maintained, this leads to changes in the metric. The density of events modifies the action and provides a route to the Einstein–Hilbert action and general relativity. Evaluating the Schwarzchild metric in classical limit shows that the perturbation in the density of possible events from the Minkowski case is proportional to the Newtonian gravitational potential.

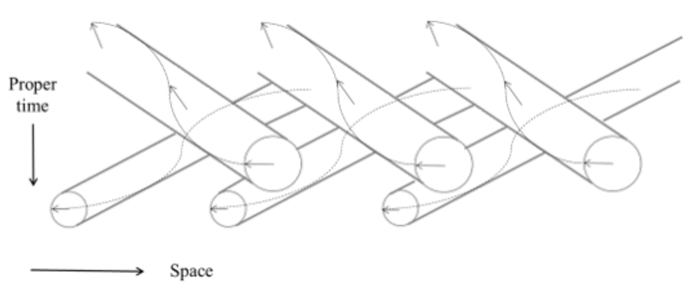

Figure 8: A physical analogy for the causal net where crossing currents correspond to spinors and the total power at each crossing vertex is the probability of observing an event.

Lastly, we can perhaps consider a common physical analogy for the causal net to aid in visualisation. Combining the net probability amplitudes in a linear manner is analogous to combining alternating or AC currents in an electrical circuit linearly to preserve phase before calculating power emitted. For the case of a free particle we previously noted that $P_{11} = P_{22}$ and $P_{12} = P_{21}$ so the causal net is analogous to electrical currents "crossing" in wires at each vertex and the probability corresponding to the total instantaneous power. If we write each individual spinor component as a current with complex phase, varying in proper time given by Eq. (19), then the sum of the real measurable apparent power from each crossing current is equivalent to the probability at the vertex. The apparent power is the modulus squared of the current multiplied by a wire resistance that is a scaling constant. This analogy is quite remarkable since, in quantum mechanics, we are usually taught to think in terms of probability currents but in fact if we think instead of the probability amplitudes being complex electrical currents and the probability being the radiated power the physical analogy is much more precise. Also interestingly the power in an electrical circuit is a conserved quantity (power in = power out) and this is also true of probability. To think of the Dirac equation as being analogous to a network of crossing electrical wires as Figure 8 is a novel representation.

5 Copyright Notice

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