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An Alternative Formulation of Einstein's Field Equations of General Relativity

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Abstract

Einstein's theory of General Relativity has stood for over 100 years now and has demonstrated more accurate predictions than Newtonian gravity; however, it is a difficult theory to use because of the non-linearity of the theory. I propose a new field equation in a flat background space-time that is analogous to the Einstein field equations, which gives the same answer as General Relativity for a point mass.

1 Introduction

Einstein's theory of General Relativity [1] has stood for over 100 years now and has demonstrated more accurate predictions than Newtonian gravity; however, it is a difficult theory to use because of the non-linearity of the theory. We present a theory of gravity in which the background metric of space-time is flat. Formulating a field equation in this flat background results in a linear theory, somewhat similar to Newton's theory. We compare the result for a point mass against the usual Schwarzschild metric and find that our theory agrees with the General Relativity result.

1.1 Flat background Metric

We base our new theory on an absolute background of flat space-time, given by the Minkowski metric $\eta_{\mu\nu}$. If using curved coordinates, we will refer to this metric as $N_{\mu\nu}$, with the knowledge that a coordinate transformation will take this metric back to the Minkowski metric. For the present article, we will use spherical coordinates (r, θ, ϕ), with radial distance r, polar angle θ and azimuthal angle ϕ. With these coordinates the flat spacetime metric becomes:

$$N_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 sin^2\theta \end{bmatrix} \tag{1}$$

Covariant differentiation using this metric will be denoted with a colon (:) so that $g_{\mu\nu:\alpha}$ indicates covariant differentiation of $g_{\mu\nu}$ using the Christoffel symbols associated with $N_{\mu\nu}$.

1.2 Gravitation Metric

We define another metric $g_{\mu\nu}$ that will represent gravity. This is the metric in Einstein's theory. This metric is interpreted differently than in General Relativity. We will not use it to lower indices or contract indices. It's sole purpose is to act on matter to influence trajectories by way of the geodesic equation. It will give the illusion of a curved space-time, but we interpret the illusion to being gravity's effect on matter, not an effect on space-time. Will [2], in reviewing Parameterized post-Newtonian formalisms of General Relativity makes the statement that linearized approximations to Einstein's theory can themselves be seen as new theories of gravity; however, metric theories of gravity (those in which matter follows geodesics with respect to $g_{\mu\nu}$ as proposed here)are typically inconsistent due to requiring zero divergence of the stress-energy tensor with respect to both the flat metric and $g_{\mu\nu}$. We get around this inconsistency by setting up a field equation(s) which does not demand flat space divergence of the stress-energy tensor. The result is a field equation more similar to the Poisson Equation for Newtonian gravity than with Einstein's Field Equations. We start with a proposed metric for a point mass:

$$g_{\mu\nu} = \begin{bmatrix} 1 - \frac{2GM}{r} & \frac{2GM}{r} & 0 & 0 \\ \frac{2GM}{r} & -1 - \frac{2GM}{r} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 sin^2\theta \end{bmatrix} \tag{2}$$

Note that this metric can be generated from the usual Schwarzschild metric $diag(1 - \frac{2GM}{r}, -1/(1 - \frac{2GM}{r}), -r^2, -r^2 sin^2\theta)$ using the Eddington-Finkelstein coordinate transformation [3,4] $dt \mapsto dt + \frac{2GM}{r-2GM}dr$. Where $\frac{2GM}{r}$ appears in $g_{\mu\nu}$, it is interpreted as a gravitational potential $\Omega_{\mu\nu}$. Every particle will generate it's own potential to be added to $N_{\mu\nu}$ to produce $g_{\mu\nu}$. We thus may sum several potentials together, such as $\Omega'_{\mu\nu} + \Omega''_{\mu\nu} + ...$ This gravitational metric will influence matter trajectories by demanding that matter follows geodesics in terms of this metric as in Einstein's theory. Covariant differentiation in terms of this metric will be the usual semicolon notation (;) as opposed to the colon (:) notation used for the flat background metric.

1.3 The Proposed Field Equations

The new proposed field equation is:

$$T' \frac{8\pi G}{c^4} = A' \tag{3}$$

We define:

$$A'_{\rho\sigma\mu\nu} = \frac{1}{2}(\Omega'_{\rho\nu:\sigma\mu} - \Omega'_{\rho\mu:\sigma\nu} + \Omega'_{\sigma\mu:\rho\nu} - \Omega'_{\sigma\nu:\rho\mu}) \tag{4}$$

and

$$A'_{\sigma\nu} = N^{\rho\mu} A'_{\rho\sigma\mu\nu} \tag{5}$$

and

$$A' = N^{\sigma\nu} N^{\rho\mu} A'_{\rho\sigma\mu\nu} \tag{6}$$

Note that $T'_{\mu\nu}$ produces $\Omega'_{\rho\nu}$ and $T''_{\mu\nu}$ produces $\Omega''_{\rho\nu}$ etc. Of note is that the following more obvious equation cannot work, because it demands zero flat space divergence of the stress-energy tensor.

$$T'_{\mu\nu} \frac{8\pi G}{c^4} = (A'_{\mu\nu} - \frac{1}{2} A' N_{\mu\nu}) \tag{7}$$

2 Discussion

We will now demonstrate that the divergence of the rhs of equation 7 is zero using covariant differentiation (:) in terms of the flat background metric $N_{\mu\nu}$, thus ruling it out as an option. We will also demonstrate that the vacuum solution for a point mass (equation 2) obeys equation 3. We will also derive the full form of Equation 3 by assuming the potential given in equation 2, using linearity and assuming the source matter is distributed as a mass density over some region.

2.1 Divergence of the RHS of Equation 7

For this demonstration we use the Mikowski metric for the background metric. Note that colon (:) differentiation is commutative since the metric is flat, and with the Minkoski metric the covariant derivatives become

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partial derivatives denoted with commas. We want to demonstrate that the following expression (rhs of the optional field equation) has zero divergence.

$$\eta^{\rho\mu}\frac{1}{2}(\Omega'_{\rho\nu,\sigma\mu}-\Omega'_{\rho\mu,\sigma\nu}+\Omega'_{\sigma\mu,\rho\nu}-\Omega'_{\sigma\nu,\rho\mu})-\eta^{\lambda\tau}\eta^{\rho\mu}\frac{1}{4}(\Omega'_{\rho\tau,\lambda\mu}-\Omega'_{\rho\mu,\lambda\tau}+\Omega'_{\lambda\mu,\rho\tau}-\Omega'_{\lambda\tau,\rho\mu})\eta_{\sigma\nu} \tag{8}$$

We differentiate using the flat background Minkowski metric in the index β and contract this with the index σ:

$$\begin{aligned}
&\eta^{\beta\sigma}\eta^{\rho\mu}\frac{1}{2}(\Omega'_{\rho\nu,\sigma\mu\beta}-\Omega'_{\rho\mu,\sigma\nu\beta}+\Omega'_{\sigma\mu,\rho\nu\beta}-\Omega'_{\sigma\nu,\rho\mu\beta})-\eta^{\beta\sigma}\eta^{\lambda\tau}\eta^{\rho\mu}\frac{1}{4}(\Omega'_{\rho\tau,\lambda\mu\beta}-\Omega'_{\rho\mu,\lambda\tau\beta}+\Omega'_{\lambda\mu,\rho\tau\beta}-\Omega'_{\lambda\tau,\rho\mu\beta})\eta_{\sigma\nu}\\
&=\eta^{\beta\sigma}\eta^{\rho\mu}\frac{1}{2}(\Omega'_{\rho\nu,\sigma\mu\beta}-\Omega'_{\rho\mu,\sigma\nu\beta}+\Omega'_{\sigma\mu,\rho\nu\beta}-\Omega'_{\sigma\nu,\rho\mu\beta})-\eta^{\beta\sigma}\eta^{\lambda\tau}\eta^{\rho\mu}\frac{1}{2}(\Omega'_{\rho\tau,\lambda\mu\beta}-\Omega'_{\rho\mu,\lambda\tau\beta})\eta_{\sigma\nu}\\
&=\eta^{\beta\sigma}\eta^{\rho\mu}\frac{1}{2}(\Omega'_{\rho\nu,\sigma\mu\beta}-\Omega'_{\rho\mu,\sigma\nu\beta}+\Omega'_{\sigma\mu,\rho\nu\beta}-\Omega'_{\sigma\nu,\rho\mu\beta})-\eta^{\lambda\tau}\eta^{\rho\mu}\frac{1}{2}(\Omega'_{\rho\tau,\lambda\mu\nu}-\Omega'_{\rho\mu,\lambda\tau\nu})\\
&=\eta^{\tau}\eta^{\rho\mu}\frac{1}{2}(\Omega'_{\rho\nu,\tau\mu}-\Omega'_{\rho\mu,\tau\nu\lambda}+\Omega'_{\tau\mu,\rho\nu}-\Omega'_{\tau\nu,\rho\mu\lambda})-\eta^{\lambda\tau}\eta^{\rho\mu}\frac{1}{2}(\Omega'_{\rho\tau,\lambda\mu\nu}-\Omega'_{\rho\mu,\lambda\tau\nu})=0
\end{aligned} \tag{9}$$

The above manipulations result from relabelling dummy indices. This demonstrates that the rhs of the optional field equations (Equation 7) has zero divergence in the flat background metric and thus is likely incompatible with a metric theory of gravity. Because of this, we seek a different equation, and propose equation 3.

2.2 Vacuum Solution

We assume that the solution given by the metric in Equation 2 complies with the vaccuum solution of our new field equation. We are using spherical coordinates for the background metric, with the non-zero Christoffel symbols given in the Appendix. The potential tensor is:

$$\Omega_{\mu\nu}=\begin{bmatrix}-\frac{2GM}{r} & \frac{2GM}{r} & 0 & 0\\ \frac{2GM}{r} & -\frac{2GM}{r} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \tag{10}$$

See the appendix, for the $A_{\rho\sigma\mu\nu}$ expanded out, using Christoffel symbols associated with the flat metric. This tensor may be readily simplified for each component by using the fact that most Christoffel symbols are zero, most of the $\Omega_{\mu\nu}$ entries are zero, and derivatives with respect to the 1st and last coordinates are zero. The result of the non-zero entries (not showing the ones that are related by symmetry) is:

$$\begin{aligned}
A_{0101}&=\frac{-\Omega_{00}}{r^2}\\
A_{0202}&=\frac{\Omega_{00}}{2}\\
A_{0303}&=\frac{\Omega_{00}sin^2\theta}{2}\\
A_{1212}&=\frac{-\Omega_{00}}{2}\\
A_{2323}&=\Omega_{00}r^2sin^2\theta\\
A_{3131}&=\frac{-\Omega_{00}sin^2\theta}{2}
\end{aligned} \tag{11}$$

Contracting the above result (using the flat metric) results in $A_{\sigma\nu}=0$, indicating it is indeed a vacuum solution.

2.3 Non-Vacuum Solution

We assume a mass that is at rest at the origin of coordinates and with mass densityρ. We then note that, using equation 10, the contracted $\Omega=0$.We may then simplify the rhs of equation 3 as:

$$A=N^{\sigma\nu}N^{\rho\mu}\Omega_{\rho\nu:\sigma\mu}=N^{\sigma\nu}B_{\nu:\sigma} \tag{12}$$

We have defined an intermediate vector $B_\nu = N^{\rho\mu}\Omega_{\rho\nu:\mu}$. We may now apply the divergence theorem on B_ν, noting that the only $\Omega_{\mu\nu}$ term to survive will be Ω_{11} and the only B_ν term that applies is $B_1 = -\frac{2GM}{r^2}$:

$$\oint_s B_\nu \cdot \hat{r} ds = 8\pi GM = \oint_V 8\pi G\rho dV = \oint_V N^{\sigma\nu} B_{\nu:\sigma} dV \tag{13}$$

Where the surface and volume integrals are over a sphere of charge density. This equation reduces to equation 3 when ρ is identified as the contracted stress-energy tensor.

3 Conclusions

We have presented a field equation analogous to the Poisson Equation for Newtonian gravity, but which agrees with Einstein's field eqautions for a point mass; however, our new proposed equation resides in a flat space-time instead of a curved space-time, so that it is linear (see equation 4). A gravitational source does not affect the background metric, and so does not affect the covariant derivative. We expect that the solution given for a point mass (equation 2) could be used with another point mass by simply adding their potentials together to get the total solution.

We are now left with a perplexing situation. The proposed field equation gives agreement with the Schwarzschild solution, but it is linear and Einstein's equations are non-linear. Einstein's field equations appear to be no less correct than our proposed field equation, but we do not know if they are compatible.

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5 References

[1]Einstein, Albert (1915), "Die Feldgleichungen der Gravitation", Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin: 844–847

[2]Will, C. M.(1973) Relativistic Gravity tn the Solar System. 111. Experimental Disproof of a Class of Linear Theories of Gravitation, Astrophysical Journal, Vol. 185, pp. 31-42

[3]Eddington, A.S. (Feb 1924). "A Comparison of Whitehead's and Einstein's Formulæ". Nature. 113 (2832): 192.

[4]Finkelstein, David (1958). "Past-Future Asymmetry of the Gravitational Field of a Point Particle". Phys. Rev. 110 (4): 965–967

6 Appendices

6.1 Christoffel Symbols for Spherical Coordinates

$$\Gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & 0 & -rsin^2\theta \end{bmatrix} \tag{14}$$

$$\Gamma^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1/r & 0 \\ 0 & 1/r & 0 & 0 \\ 0 & 0 & 0 & -sin\theta cos\theta \end{bmatrix} \tag{15}$$

$$\Gamma^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/r \\ 0 & 0 & 0 & cos\theta/sin\theta \\ 0 & 1/r & cos\theta/sin\theta & 0 \end{bmatrix} \tag{16}$$

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6.2 The "A" Tensor

$$\begin{aligned}
2A_{\rho\sigma\mu\nu} = {} & \Omega_{\rho\nu,\sigma\mu} - \Omega_{\rho\lambda,\mu}\Gamma^{\lambda}_{\nu\sigma} - \Omega_{\rho\lambda}\Gamma^{\lambda}_{\nu\sigma,\mu} - (\Omega_{\eta\nu,\sigma} - \Omega_{\eta\lambda}\Gamma^{\lambda}_{\nu\sigma})\Gamma^{\eta}_{\rho\mu} - (\Omega_{\rho\nu,\eta} - \Omega_{\rho\lambda}\Gamma^{\lambda}_{\nu\eta})\Gamma^{\eta}_{\sigma\mu} \\
& -\Omega_{\rho\mu,\sigma\nu} + \Omega_{\rho\lambda,\nu}\Gamma^{\lambda}_{\mu\sigma} + \Omega_{\rho\lambda}\Gamma^{\lambda}_{\mu\sigma,\nu} + (\Omega_{\eta\mu,\sigma} - \Omega_{\eta\lambda}\Gamma^{\lambda}_{\mu\sigma})\Gamma^{\eta}_{\rho\nu} + (\Omega_{\rho\mu,\eta} - \Omega_{\rho\lambda}\Gamma^{\lambda}_{\mu\eta})\Gamma^{\eta}_{\sigma\nu} \\
& +\Omega_{\sigma\mu,\rho\nu} - \Omega_{\sigma\lambda,\nu}\Gamma^{\lambda}_{\mu\rho} - \Omega_{\sigma\lambda}\Gamma^{\lambda}_{\mu\rho,\nu} - (\Omega_{\eta\mu,\rho} - \Omega_{\eta\lambda}\Gamma^{\lambda}_{\mu\rho})\Gamma^{\eta}_{\sigma\nu} - (\Omega_{\sigma\mu,\eta} - \Omega_{\sigma\lambda}\Gamma^{\lambda}_{\mu\eta})\Gamma^{\eta}_{\rho\nu} \\
& -\Omega_{\sigma\nu,\rho\mu} + \Omega_{\sigma\lambda,\mu}\Gamma^{\lambda}_{\nu\rho} + \Omega_{\sigma\lambda}\Gamma^{\lambda}_{\nu\rho,\mu} + (\Omega_{\eta\nu,\rho} - \Omega_{\eta\lambda}\Gamma^{\lambda}_{\nu\rho})\Gamma^{\eta}_{\sigma\mu} + (\Omega_{\sigma\nu,\eta} - \Omega_{\sigma\lambda}\Gamma^{\lambda}_{\nu\eta})\Gamma^{\eta}_{\rho\mu}
\end{aligned} \tag{17}$$

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