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Binding Energy Across Classical Black Holes: A Horizon-Based Thermodynamic Approach

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Abstract

We present analytic expressions for the dimensionful quantity

$$\text{B.E.} = \kappa M r_+,$$

which combines surface gravity κ, black hole mass M, and event-horizon radius r_+, across classical black-hole spacetimes: Kerr–Newman (rotating, charged), Kerr (rotating), Reissner–Nordström (charged), and Schwarzschild (neutral, non-rotating). This expression serves as a thermodynamic energy proxy, reflecting the interplay between horizon properties and gravitational binding in black holes. It recovers the classical result $\frac{1}{2}Mc^2$ for the Schwarzschild case and vanishes in extremal limits—consistent with the third law of black hole thermodynamics. We further discuss implications for gravitational energy extraction and astrophysical jet formation.

1 Introduction

The profound analogy between gravity and thermodynamics is manifest in black hole physics, where quantities like surface gravity κ and horizon area take on the roles of temperature and entropy, respectively [1,2]. A longstanding question is how to characterize energy availability or gravitational binding near a black hole's event horizon.

We introduce a simple yet physically motivated quantity:

$$\text{B.E.} = \kappa M r_+, \tag{1}$$

which has dimensions of energy. While not the canonical "binding energy" derived from asymptotic mass differences, this expression captures how local geometric and thermodynamic quantities contribute to energy scales relevant for processes like accretion and jet formation. We demonstrate that this proxy vanishes in extremal limits and interpolates smoothly across classical solutions.

This approach complements existing frameworks based on irreducible mass [3], Penrose processes [4], and Blandford–Znajek energy extraction [5], offering a thermodynamically grounded estimator of energy potential near the event horizon.

2 Formulation

2.1 Kerr–Newman Metric

The Kerr–Newman black hole is described by its mass M, charge Q, and specific angular momentum $a = J/M$. The outer horizon radius is:

$$r_+ = \frac{GM}{c^2} + \delta, \quad \delta = \sqrt{\left(\frac{GM}{c^2}\right)^2 - a^2 - \frac{GQ^2}{4\pi\varepsilon_0 c^4}}. \tag{2}$$

The surface gravity is given by:

$$\kappa = \frac{c^2\delta}{r_+^2 + a^2}. \tag{3}$$

Using the identity:

$$r_+^2 + a^2 = 2\left(\frac{GM}{c^2}\right)^2 + 2\frac{GM}{c^2}\delta - \frac{GQ^2}{4\pi\varepsilon_0 c^4}, \tag{4}$$

we obtain the expression for the binding energy proxy:

$$\text{B.E.}_{\text{KN}} = \frac{M\delta\left(\frac{GM}{c^2} + \delta\right)c^2}{2\left(\frac{GM}{c^2}\right)^2 + 2\frac{GM}{c^2}\delta - \frac{GQ^2}{4\pi\varepsilon_0 c^4}}. \tag{5}$$

This result reduces smoothly to Kerr or Reissner–Nordström when $Q \to 0$ or $a \to 0$, respectively.

2.2 Special Cases

Kerr (Q = 0)

$$\delta = \sqrt{\left(\frac{GM}{c^2}\right)^2 - a^2}, \quad \text{B.E.}_{\text{Kerr}} = \frac{Mc^2}{2} \cdot \frac{\delta}{GM/c^2}. \tag{6}$$

As expected, rotation reduces the available binding energy compared to Schwarzschild.

Reissner–Nordström (a = 0)

$$\delta = \sqrt{\left(\frac{GM}{c^2}\right)^2 - \frac{GQ^2}{4\pi\varepsilon_0 c^4}}, \quad \text{B.E.}_{\text{RN}} = \frac{Mc^2\delta\left(\frac{GM}{c^2} + \delta\right)}{2\left(\frac{GM}{c^2}\right)^2 + 2\frac{GM}{c^2}\delta - \frac{GQ^2}{4\pi\varepsilon_0 c^4}}. \tag{7}$$

Charge likewise diminishes the binding energy.

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Schwarzschild (Q = 0, a = 0)

$$r_+ = \frac{2GM}{c^2}, \quad \kappa = \frac{c^4}{4GM}, \quad \text{B.E.}_S = \frac{1}{2}Mc^2. \tag{8}$$

Extremal Limits

For extremal Kerr ($a = GM/c^2$) or extremal RN ($Q^2 = 4\pi\varepsilon_0 GM^2c^2$), we find $\delta = 0$, implying $\kappa = 0$ and hence B.E. = 0. This result aligns with the third law of black hole thermodynamics.

3 Results and Observations

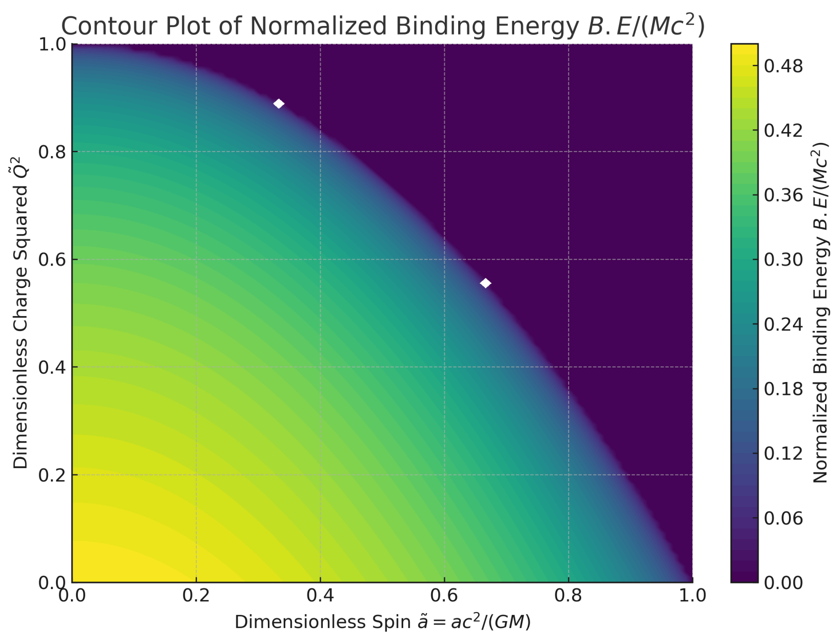

Figure 1: Contour plot of normalized binding energy $B.E/(Mc^2)$ with respect to dimensionless spin $\tilde{a}$ and charge squared $\tilde{Q}^2$. Energy decreases as black holes approach extremality.

We define dimensionless parameters:

$$\tilde{a} = \frac{ac^2}{GM}, \quad \tilde{Q}^2 = \frac{Q^2G}{4\pi\varepsilon_0 G^2M^2c^2}. \tag{9}$$

The normalized binding energy B.E./(Mc^2) exhibits the following behavior:

- **Monotonic Decrease**: It decreases with increasing $\tilde{a}$ and $\tilde{Q}^2$.
- **Extremal Limit**: As $\tilde{a}^2 + \tilde{Q}^2 \to 1$, B.E. $\to 0$.
- **Energetic Implications**: Spin and charge reduce the gravitational energy available for physical processes, including jet launching and accretion heating.

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- **Thermodynamic Insight**: Lower surface gravity correlates with lower binding energy, reinforcing the interpretation of κ as a temperature proxy.

4 Discussion

The proxy B.E. $= \kappa M r_+$ encodes both geometric and thermodynamic properties at the black hole horizon. While not a substitute for global mass-energy definitions (e.g., ADM or Komar mass), it provides an intuitive handle on the *extractable* or *interacting* energy scale close to the horizon.

In comparison:

- The **irreducible mass** formalism [3] identifies energy extractable via spin or charge.
- The **Penrose process** [4] exploits the ergosphere's frame-dragging for energy extraction.
- The **Blandford–Znajek mechanism** [5] relies on magnetohydrodynamic interactions with spinning black holes.

This proxy aligns with those paradigms by predicting reduced energy availability as black holes approach extremality.

5 Conclusion

We have proposed and analyzed a simple expression B.E. $= \kappa M r_+$ as a thermodynamically motivated proxy for gravitational binding energy across classical black hole spacetimes. This framework:

- Reproduces known results in the Schwarzschild case,
- Reduces smoothly across Kerr, RN, and Kerr–Newman metrics,
- Vanishes in extremal limits,
- Offers physical insights into astrophysical black holes and energetic phenomena.

Future work could explore its application in modified gravity theories, numerical relativity, or observational constraints from accretion disks and jet efficiencies.

6 Copyright Notice

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