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Homogeneous Turbulence : A New Study

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Abstract

Navier-Stokes equations is studied for statistically homoegeneous turbulence in incompressible fluid flow. In this paper the inertial transfer term is studied in a novel way. A solution of hitherto unsolvable NV Stokes equations has been obtained.

1 Introduction

Turbulence has been studied by scientists for about a century. Many path break- ing works have been done including quaternionic application. It is not possible to name all the scientists. But the names of Ladyzhanskya, Kolmogorov, Tay- lor, Batchelor, Kraichnan, and many others may be mentioned [1-18].

Here we have taken a different path. The combination of operation calculus, and dimensional analysis is used. Now let us start with the Navier- Stokes Equations for incompressible flow with zero mean velocity. It is also assumed that pressure gradient is negligible. At the point (x), and time t the equation is given along with the continuity equation.

$$\begin{aligned} \frac{\delta u_i}{\delta t} + u_k \frac{\delta u_i}{\delta x_k} &= \nu \frac{\delta^2 u_i}{\delta x_k \delta x_k} \\ \frac{\delta u_i}{\delta x_i} &= 0 \end{aligned} \tag{1}$$

Similarly at the second point x', but at the same time we have the equation as

$$\begin{aligned} \frac{\delta u'_j}{\delta t} + u'_k \frac{\delta u'_j}{\delta x'_k} &= \nu \frac{\delta^2 u'_j}{\delta x'_k \delta x'_k} \\ \frac{\delta u'_i}{\delta x'_i} &= 0 \end{aligned} \tag{2}$$

1.1 Homogeneity

$\vec{r} = \vec{x'} - \vec{x}$. Hence

$$\frac{\delta}{\delta r_i} = \frac{\delta}{\delta x'_i} = -\frac{\delta}{\delta x_i}$$

.

Multiplying eq(1) by u'_j, and eq(2) by u_i, and then taking ensemble average, and using rules of homogeneity, and incompressibility as above we get

$$\frac{\delta\overline{u_i u'_j}}{\delta t} + \overline{u_k \frac{u_i u'_j}{\delta x_k}} + \overline{u'_k \frac{u_i u'_j}{\delta x'_k}} = 2\nu \frac{\delta^2 \overline{u_i u'_j}}{\delta r_k \delta r_k} \tag{3}$$

The second, and third term of the last equation is the main hindrance to the solvability of this equation. Before studying the solvability problem let us examine the stages of turbulence.

2 Stages of Turbulence

2.1 Stage 1

In this stage we envisage that there are large eddies of different shapes, though not very different from sphere. The diameters of these eddies in Cartesian Coordinate may be L_k, for $(k = 1, 2, 3)$. The average diameters may be $\overline{L_k}$.These eddies gradually break down into smaller eddies. These eddies may gradually cascade to smaller, and smaller eddies to reach the end of this stage. During the cascading process eddies become more, and more spherical. The smallest eddies may be perfectly spherical, and the size of the smallest eddies may be same as Taylor's microscale ℓ. In this stage there may not be any perceptible mixing. Only the cascading eddies are statistically correlated. We call this stage Correlation stage.

Average cascading velocity may be initially $-v_{1k} = -\frac{\delta\overline{L_k}}{\delta t}$, and finally $-v_{2k} = -\frac{\delta \ell_k}{\delta t}$. The weighted average of cascading velocity may be given as

$$-U_k = -(\alpha v_{1k} + (1 - \alpha) v_{2k}) \tag{4}$$

But how can we define α? We observe that during the cascading process Reynold's number related to the largest eddies is given as $Re_{|\overline{L_k}|} = \frac{|\overline{L_k}||v_{1k}|}{\nu}$. Similarlly that related to Taylor's microscale may be given as $Re_\ell = \frac{\ell |v_{2k}|}{\nu}$. Reynold's number changes along with eddy size. hence we define $\alpha = \frac{\ell^2}{|\overline{L_k}|^2}$.

2.2 Stage 2

In stage two mixing may be the main process. Eddies mix with one another. This process starts from eddies of Taylor's microscale, and ends at Kolmogorov microscale τ. From dimensional reasoning speed starts from $\Omega\ell$, and ends at $\Omega\tau$. Since it is a short range, Ω may be taken unchanged over the range, and the simple average of these two speeds may also be be taken as

$$-V_k = -0.5((\Omega\ell)_k + (\Omega\tau)_k) \tag{5}$$

2.3 Combined Stage

$$-(U_k + V_k) = -(\alpha v_{1k} + (1 - \alpha) v_{2k}) - 0.5((\Omega\ell)_k + (\Omega\tau)_k) \tag{6}$$

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3 Main Equation

We have already said that the second, and the third terms of the equation are the main terms to be tackled for the solution of the equation. Let us go back to the Combined equation again.

$$-(U_k + V_k) = -(\alpha v_{1k} + (1-\alpha)v_{2k}) - 0.5((\Omega\ell)_k + (\Omega\tau)_k) \tag{7}$$

The distance between two probes is $\vec{r}$. Now we suggest that the inertial transfer may be repleced by

$$-(U_k + V_k)\frac{\delta}{\delta r_k} = -((\alpha v_{1k} + (1-\alpha)v_{2k}) - 0.5((\Omega\ell)_k + (\Omega\tau)_k)\frac{\delta}{\delta r_k} \tag{8}$$

The main equation then becomes

$$\frac{\delta \overline{u_i u_j'}}{\delta t} - ((\alpha v_{1k} + (1-\alpha)v_{2k}) - 0.5((\Omega\ell)_k + (\Omega\tau)_k)\frac{\delta}{\delta r_k}\overline{u_i u_j'} = 2\nu \frac{\delta^2 \overline{u_i u_j'}}{\delta r_k \delta r_k} \tag{9}$$

It may be noted that the seggested replacement of inertial transfer operator is already an averaged one. Hence it is uncorrelated to the $\overline{u_i u_j'}$.

3.1 Steady State Equation

The steady state equation is given as

$$\frac{\delta^2 \overline{u_i u_j'}}{\delta r_k \delta r_k} + P_k \frac{\delta}{\delta r_k}\overline{u_i u_j'} = 0 \tag{10}$$

where $P_k = \frac{((\alpha v_{1k} + (1-\alpha)v_{2k}) - 0.5((\Omega\ell)_k + (\Omega\tau)_k)}{2\nu}$.

4 Different Symmetries

Turbulence is generally studied for different statistical symmetries. In the present paper we would discuss only spherical symmetry. The reason behind generation of this symmetry is to be examined first. It has already been envisaged that turbulence is created by some system of hindrance to the flow of fluid. It immediately establishes one fact that the seed of a chosen symmetry is embedded from the very begining. But we have divided the course of turbulence into two stages prior to the final decay stage. In the first stage cascading process is considered; where in the second stage mixing is the dominant process.

4.1 Spherical Symmetry

We envisage that in case of spherical symmetry eddies not only become smaller, and smaller, in the first stage, but also more, and more spherical. Probably this act of transformation is due to grinding action, and also due to initial system that produces the turbulence. The transformation of eddies into perfect sphere is completed in stage two. Then the eddies would be perfectly spherical. In fact $P_k \frac{\delta}{\delta r_k}$ would become $3 \times P_1 \frac{\delta}{\delta r_1}$, or simply $3 \times P \frac{\delta}{\delta r}$, where r is the distance of the two probes. Accordingly eq(11) becomes

$$\frac{\delta^2 \overline{u_i u_j'}}{\delta r \delta r} + P\frac{\delta}{\delta r}\overline{u_i u_j'} = 0 \tag{11}$$

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For a solution of eq(11) let us take $i = j = 1$. Then eq(11) becomes

$$\frac{\delta^2 \overline{u_1 u_1'}}{\delta r \delta r} + P \frac{\delta}{\delta r} \overline{u_1 u_1'} = 0 \tag{12}$$

Now let us put $\overline{u_1 u_1'} = \overline{u_1^2} f$, where $\overline{u_1^2} = \overline{u_2^2} = \overline{u_3^2} = \frac{1}{3}\overline{u^2}$. So we have

$$\frac{\delta^2 f}{\delta r \delta r} + P \frac{\delta f}{\delta r} = 0 \tag{13}$$

Or

$$\frac{\delta f}{\delta r} + Pf = C_1 \tag{14}$$

where C_1 is a constant of integration. With the boundary conditions $r = 0$ $f = 1$, and $r = \ell$ $f --> 0$, we have

$$f = 1 - r(\frac{1}{\ell} + \frac{P}{\nu}\ell) + \frac{P}{\nu} r^2 \tag{15}$$

5 Comment

In this paper we have taken a novel path to solve the NV equations in statistically homogeneous condition. A solution has been given for a simplified form. The process adopted here lies in the fundamental concept of turbulent flow. The concept is not new. It has been followed by many prominant mathematicians. We have followed their footstep only differing in the approach. In doing so we have made several assumptions. The first assumption is that there is one zone for cascading of eddies from bigger to smaller eddies only. The second assumption is that in the other zone there is mixing of eddies. It has been argued that these zones are inherent property of a particular turbulent flow, and uncorrelated to the measurement results of the probes.

Here we have studied NV-equations for spherical symmetry only. It is expected that the process may be extended to different types of symmetries, including isotropy, axisymmetry, etc. However, in each case eq(9) is the starting point.

6 Copyright Notice

7 Conflict of Interest

The author hereby declares that nothing has been borrowed from any source. There is no conflict of interest.

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