

Parametric Control of Two-Mode Squeezed Light via Engineered Reservoirs and Tunable Optomechanical Interfaces

Tarekegn Ossie¹, Misrak Getahun²

Department of Physics, Hawassa University, P. O. Box 05, Hawassa, Ethiopia

¹tarekegnossie876@gmail.com, ²misrakg@hu.edu

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Abstract

Quantum optomechanical systems enable versatile control of non-classical light. We study two-mode squeezed-light generation in a driven optomechanical cavity with parametrically amplified optical modes coupled to mechanical resonators. Coherent driving reveals a rich control landscape where parametric nonlinearity, optomechanical coupling, and reservoir interactions jointly tailor quantum correlations-including two-mode squeezing, entanglement, steering, and discord. Remarkably, mechanical oscillators not only mediate optical entanglement but also act as a squeezing resource, enabling enhancement or suppression of quantum correlations via parameter tuning. We derive steady-state second-order moments and closed-form covariance matrix expressions, quantifying entanglement via logarithmic negativity, steering via Reid criterion, and discord via von Neumann entropy, providing a comprehensive framework for optimized generation of quantum correlations in hybrid quantum systems.

1 Introduction

The generation and control of non-classical states of light are foundational goals in quantum optics, with far-reaching implications for quantum information processing, precision metrology, and fundamental tests of quantum mechanics [16, 28]. Among these, two-mode squeezed light-the optical realization of Einstein-Podolsky-Rosen (EPR) entangled states [34]-has emerged as a critical resource for quantum technologies, enabling quantum teleportation [7], entanglement-based cryptography [37], and interferometric measurements beyond the standard quantum limit [12, 38].

Traditional generation of two-mode squeezed light relies on parametric processes in nonlinear crystals [40, 53, 57] or four-wave mixing in atomic ensembles [3, 31]. While successful, these methods offer limited tunability and face integration challenges. Cavity optomechanics-where mechanical motion couples parametrically to optical fields [1, 23]-has introduced transformative paradigms for quantum state engineering. Mechanical elements provide additional quantum degrees of freedom and enable dynamic control via radiation pressure [1, 2, 6], offering unique advantages: mechanical oscillators mediate optical correlations [15, 47], coupling strengths are tunable *in situ* [1], and integrated platforms enable scalable implementations [9, 13, 39, 44]. Experimental milestones include ground-state cooling of nanomechanical oscillators [9, 43], observation of quantum motion [39], and quantum-coherent optomechanical coupling [17, 45].

The integration of parametric squeezing with optomechanics introduces complex dynamics where quantum noise, thermal fluctuations, and coherent driving compete [12]. Coherent drives

pump the parametric process while controlling the mechanical state, enabling ground-state cooling [15, 30, 51] and optical squeezing [8, 35, 52], making driven optomechanical systems attractive for generating two-mode squeezed light [26, 33]. Recent advances show that parametric driving enhances squeezing bandwidth and reduces quantum noise [10, 50, 56], while squeezed vacuum reservoirs suppress noise beyond conventional limits. This hybrid architecture facilitates broadband operation via combined optical spring and parametric effects, enabling quantum-enhanced sensing and efficient entanglement generation [4, 22, 27, 48].

The system comprises a nondegenerate parametric oscillator embedded within an optomechanical cavity, where two distinct optical modes (signal and idler) are parametrically coupled. This configuration enables two-mode squeezing [34, 57], frequency-multiplexed entanglement, mode-selective quantum control, EPR steering [38], and quantum transduction [4, 22]. The cavity is coherently driven and coupled to a squeezed vacuum reservoir for noise suppression, while the mechanical mode is connected to a thermal bath ($\bar{n}$) inducing controlled decoherence [15, 30, 51]. We present a comprehensive theoretical investigation of squeezed light and quantum correlations in this nondegenerate parametric oscillator-optomechanical hybrid system.

2 Cavity Mode Variables

We study a nondegenerate parametric oscillator in which two optical modes (signal and idler) are parametrically coupled inside an optomechanical cavity and interact with a mechanical oscillator via radiation pressure. The dynamics-parametric amplification, optomechanical coupling, coherent driving, and reservoir interactions-are captured by the Hamiltonian. Working in the interaction picture with respect to the cavity free energy $\hbar\omega_c \sum_j \hat{a}_j^\dagger \hat{a}_j$ (hence omitted) while retaining the mechanical free energy $\hbar\omega_m \sum_j \hat{b}_j^\dagger \hat{b}_j$, we have:

$$\begin{aligned} \hat{H} = & \hbar\omega_m \sum_{j=1}^2 \hat{b}_j^\dagger \hat{b}_j + i\hbar \sum_{j=1}^2 \varepsilon_j (\hat{a}_j^\dagger - \hat{a}_j) \\ & + \hbar \sum_{j=1}^2 g_j [\alpha_j^* \hat{a}_j + \alpha_j \hat{a}_j^\dagger] (\hat{b}_j + \hat{b}_j^\dagger) \\ & + i\hbar \chi (\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1 \hat{a}_2) \\ & + i\hbar \sum_{j,k} \left[\lambda_{jk} (\hat{a}_j^\dagger \hat{A}_{jk} - \hat{A}_{jk}^\dagger \hat{a}_j) + \eta_{jk} (\hat{b}_j^\dagger \hat{B}_{jk} - \hat{B}_{jk}^\dagger \hat{b}_j) \right], \end{aligned} \quad (1)$$

where ε_j is the driving amplitude, g_j the optomechanical coupling strengths, α_j the classical amplitude, χ the two-mode squeezing (parametric down-conversion) strength, and λ_{jk}, η_{jk} coupling constants to external squeezed vacuum and thermal baths $\hat{A}_{jk}, \hat{B}_{jk}$. The master equation is:

$$\begin{aligned} \frac{d\rho}{dt} = & -\frac{i}{\hbar} [\hat{H}_S, \rho] \\ & + \sum_{j=1}^2 \frac{\kappa_j}{2} \left[(N_j + 1) \mathcal{D}[\hat{a}_j] \rho + N_j \mathcal{D}[\hat{a}_j^\dagger] \rho \right. \\ & \left. - M_j \mathcal{S}[\hat{a}_j] \rho - M_j^* \mathcal{S}[\hat{a}_j^\dagger] \rho \right] \\ & + \sum_{j=1}^2 \frac{\gamma_j}{2} \left[(\bar{n}_j + 1) \mathcal{D}[\hat{b}_j] \rho + \bar{n}_j \mathcal{D}[\hat{b}_j^\dagger] \rho \right], \end{aligned} \quad (2)$$

with $\hat{H}_S$ containing the coherent, optomechanical, and parametric terms from Eq. (1), and super-operators:

$$\mathcal{D}[\hat{O}]\rho = 2\hat{O}\rho\hat{O}^\dagger - \hat{O}^\dagger\hat{O}\rho - \rho\hat{O}^\dagger\hat{O}, \quad \mathcal{S}[\hat{O}]\rho = 2\hat{O}\rho\hat{O} - \hat{O}^2\rho - \rho\hat{O}^2, \quad (3)$$

where κ_j are cavity decay rates with squeezed reservoirs, $N_j = \sinh^2 r_j$ (photon number from squeezing), $M_j = \sqrt{N_j(1+N_j)}$ (squeezing correlation), γ_j mechanical damping rates, and $\bar{n}_j$ thermal bath occupations. From Eq. (2) and the Heisenberg-Langevin approach $\frac{d\langle\hat{A}\rangle}{dt} = \text{Tr}(\frac{d\rho}{dt}\hat{A})$, the steady-state second-order moments are obtained by setting time derivatives to zero:

$$A_{jk} = \frac{2}{(\kappa_j + \kappa_k)} \left[\varepsilon_k \alpha_j - \varepsilon_j \alpha_k - ig_k \alpha_k (E_{jk}^* + F_{jk}^*) + ig_j \alpha_j (E_{kj} + F_{kj}) + \chi(C_{j,3-k}^* - A_{3-j,k}) \right], \quad (4)$$

$$C_{jk} = \frac{2}{(\kappa_j + \kappa_k)} \left[\varepsilon_k \alpha_j + \varepsilon_j \alpha_k - ig_k \alpha_k (E_{jk} + F_{jk}) - ig_j \alpha_j (E_{kj} + F_{kj}) + \chi(A_{j,3-k}^* + A_{3-j,k}^*) \right], \quad (5)$$

$$B_{jk} = \frac{2}{(\gamma_j + \gamma_k)} \left[ig_j \alpha_j (E_{jk}^* + F_{jk}^*) - ig_k \alpha_k (E_{jk} + F_{jk}) \right], \quad (6)$$

$$D_{jk} = \frac{2}{(\gamma_j + \gamma_k)} \left[-ig_k \alpha_k (E_{jk}^* + F_{jk}^*) - ig_j \alpha_j (E_{jk}^* + F_{jk}^*) \right], \quad (7)$$

$$E_{jk} = \frac{2}{(\kappa_j + \gamma_k)} \left[-ig_k \alpha_k (C_{jk} + A_{jk}^*) + \varepsilon_j b_k + ig_j \alpha_j E_{jk}^* - \chi[E_{3-j,k} + F_{3-j,k} - E_{k,3-j}^*] \right], \quad (8)$$

$$F_{jk} = \frac{2}{(\kappa_j + \gamma_k)} \left[-ig_k \alpha_k (A_{jk} + C_{jk}) + \varepsilon_j b_k - ig_j \alpha_j E_{jk}^* + \chi E_{3-j,k} \right], \quad (9)$$

with $A_{jk} = \langle \hat{a}_j^\dagger \hat{a}_k \rangle$, $C_{jk} = \langle \hat{a}_j \hat{a}_k \rangle$, $B_{jk} = \langle \hat{b}_j^\dagger \hat{b}_k \rangle$, $D_{jk} = \langle \hat{b}_j \hat{b}_k \rangle$, $E_{jk} = \langle \hat{a}_j^\dagger \hat{b}_k \rangle$, $F_{jk} = \langle \hat{a}_j \hat{b}_k \rangle$. The steady-state second-order correlations are compactly expressed in terms of the moments as:

$$X_{jk}^\pm = \frac{1}{2} (A_{jk} + A_{kj} \pm (C_{jk} + C_{jk}^*)), \quad (10)$$

$$Y_{jk}^\pm = \frac{1}{2} (B_{jk} + B_{kj} \pm (D_{jk} + D_{jk}^*)), \quad (11)$$

$$Z_{jk}^\pm = \frac{1}{2} (E_{jk} + E_{kj}^* \pm (F_{jk} + F_{jk}^*)). \quad (12)$$

These symmetric and antisymmetric combinations directly correspond to quadrature covariances and are essential for evaluating the output spectrum and quantum correlations.

3 Quadrature Fluctuations and Two-Mode Squeezing

Squeezing properties are quantified via quadrature variances derived from the second-order moments $X_{jk}^\pm$, $Y_{jk}^\pm$, and $Z_{jk}^\pm$ with emphasis on two-mode variances characterizing EPR-type correlations [16, 34, 38, 49].

3.1 Two-Mode Quadrature Variances for cavity Modes

For two optical modes j and k , we define the EPR-type joint quadrature operators as

$$\hat{X}_{jk}^{\pm} = \frac{\hat{X}_j \pm \hat{X}_k}{\sqrt{2}}, \quad (13)$$

where $\hat{X}_j$ and $\hat{X}_k$ are the position quadratures of the respective modes. The variances of these operators are given by Using the correlation matrices A_{jk} and C_{jk} from Eqs. (4), and (5), the quadrature variances in terms of symmetrized moments are expressed:

$$\begin{aligned} \Delta X_{jk}^{\pm 2} = & \frac{1}{2} \left(X_{jj}^+ + X_{kk}^+ \pm 2X_{jk}^+ \right. \\ & \left. + 1 - \frac{1}{2} [(\alpha_j + \alpha_j^*) \pm (\alpha_k + \alpha_k^*)]^2 \right), \end{aligned} \quad (14)$$

where

$$X_{jj}^+ = \frac{4\varepsilon_j^2}{\kappa_j^2} + \frac{\chi}{\kappa_j} (C_{j,3-j}^* + A_{j,3-j}), \quad (15)$$

$$X_{kk}^+ = \frac{4\varepsilon_k^2}{\kappa_k^2} + \frac{\chi}{\kappa_k} (C_{k,3-k}^* + A_{k,3-k}), \quad (16)$$

$$X_{jk}^+ = \frac{8\varepsilon_j\varepsilon_k}{\kappa_j(\kappa_j + \kappa_k)} + \frac{2\chi}{\kappa_j + \kappa_k} (A_{jj}^* + C_{jj}^*) \quad (17)$$

with

$$A_{jj} = -\frac{2g_j\alpha_j}{\kappa_j} I_j + \frac{\chi}{\kappa_j} (C_{jk}^* - A_{kj}), \quad (18)$$

$$C_{jj} = \alpha_j^2 + \frac{2g_j\alpha_j}{\kappa_j} I_j + \frac{\chi}{\kappa_j} (A_{jk}^* + A_{kj}^*), \quad (19)$$

$$\begin{aligned} E_{jj} = & \frac{2}{\kappa_j + \gamma_j} \left[-ig_j\alpha_j(C_{jj} + A_{jj}^*) + \varepsilon_j b_j \right. \\ & \left. + ig_j\alpha_j E_{jj}^* - \chi(E_{3-j,j} + F_{3-j,j} - E_{j,3-j}^*) \right], \end{aligned} \quad (20)$$

$$\begin{aligned} F_{jj} = & \frac{2}{\kappa_j + \gamma_j} \left[-ig_j\alpha_j(A_{jj} + C_{jj}) + \varepsilon_j b_j \right. \\ & \left. - ig_j\alpha_j E_{jj}^* + \chi E_{3-j,j} \right], \end{aligned} \quad (21)$$

together with $I_j = \text{Im}(E_{jj} + F_{jj})$ and

$I_{jk} = \text{Im}(E_{jk} + F_{jk})$. For each mode j , the steady-state amplitudes α_j satisfies

$$\alpha_j = \frac{\varepsilon_j \left(\frac{\kappa_j}{2} - i\chi\alpha_{3-j} \right)}{\left(\frac{\kappa_j}{2} \right)^2 + \chi^2 \alpha_{3-j}^2} \quad (22)$$

where α_{3-j} denotes the amplitude of the other mode (k when j is considered). The mechanical displacement b_j by the radiation-pressure interaction is given by:

$$b_j = \frac{2ig_j}{\gamma_j} |\alpha_j|^2 \quad (23)$$

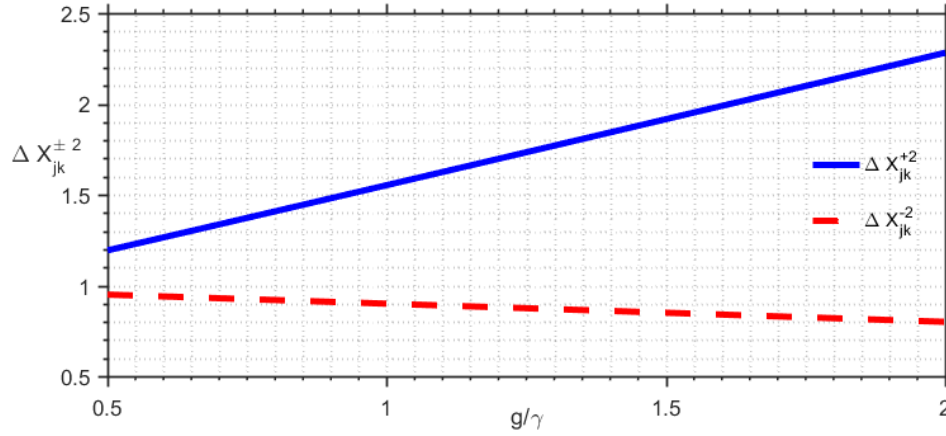


Figure 1: Plots of [Eq. (14)] versus g/γ for $\gamma = 0.1$, $\kappa = 0.8$, $\epsilon = 0.35$, and $\chi = 0.15$

FIG. 1 shows the EPR quadrature variances ΔX_{jk}^{+2} (blue) and ΔX_{jk}^{-2} (red) versus g/γ . ΔX_{jk}^{-2} exhibits squeezing (minimum 0.703, 15% below SQL), while ΔX_{jk}^{+2} shows anti-squeezing (2.369, 30% above SQL), preserving the Heisenberg uncertainty relation. Squeezing persists over $0.5 \lesssim g/\gamma \lesssim 2$, with a broad plateau indicating robust entanglement. Both variances saturate at high coupling as nonlinearities balance the drive, with maximum squeezing at intermediate coupling. These results demonstrate efficient two-mode squeezed state generation via Kerr nonlinearity, violating the classical noise limit over a wide parameter region. FIGS. 2 and 3 reveal the EPR difference

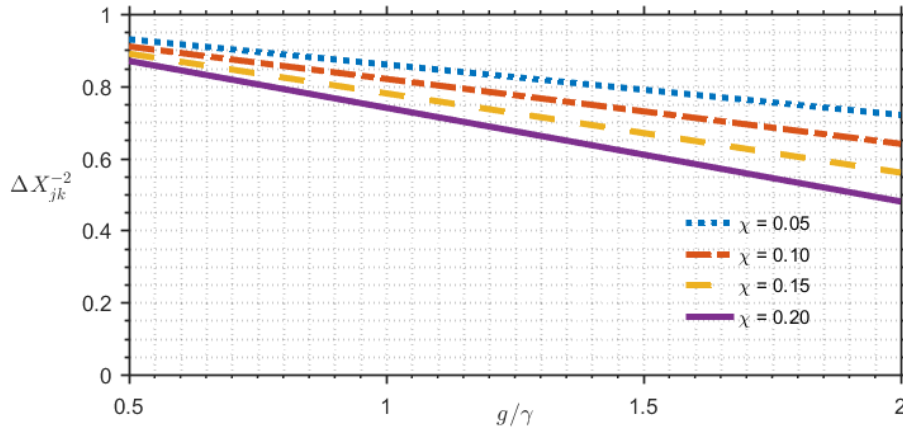


Figure 2: Plots of ΔX_{jk}^{-2} versus g/γ for $\gamma = 0.1$, $\kappa = 0.8$, $\epsilon = 0.35$, and for different χ values

variance ΔX_{jk}^{-2} versus g/γ . Fig. 2 shows that larger Kerr nonlinearity χ produces deeper squeezing (lower minima) and slightly shifts optimal coupling, enhancing inter-mode correlations. Fig. 3 shows that larger drive amplitude ϵ (with $\epsilon > 0$) produces deeper squeezing by strengthening the intracavity field and cross-Kerr interaction; no squeezing occurs for $\epsilon = 0$. Both curves saturate at high coupling as nonlinearities balance the drive. Thus, χ and ϵ are key control parameters for generating strong, tunable two-mode squeezed states.

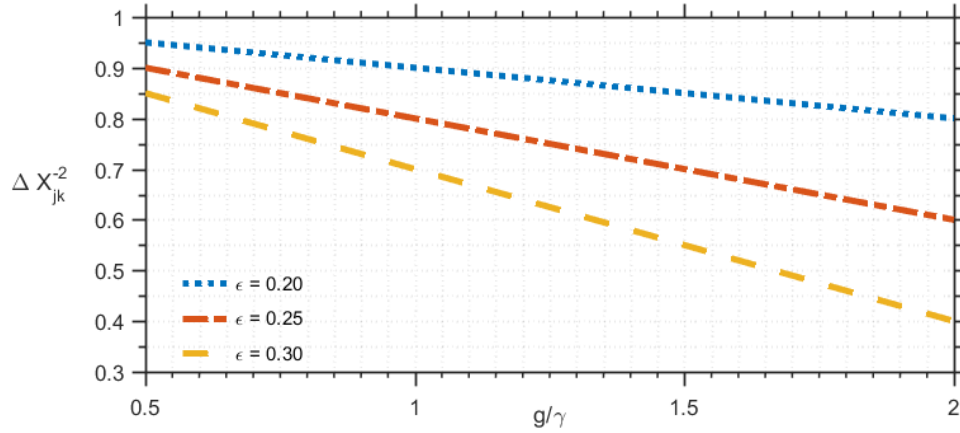


Figure 3: Plots of ΔX_{jk}^{-2} versus g/γ for $\gamma = 0.1$, $\kappa = 1.0$, $\chi = 0.15$, and for different ϵ values

For two mechanical modes j and k , the squeezing properties are characterized by the variances of the EPR-type joint quadrature operators

$$\hat{Q}_{jk}^- = \frac{\hat{Y}_j - \hat{Y}_k}{\sqrt{2}}, \quad \hat{\Pi}_{jk}^+ = \frac{\hat{Y}_j + \hat{Y}_k}{\sqrt{2}}, \quad (24)$$

where $\hat{Y}_j$ and $\hat{Y}_k$ denote the position quadratures of the mechanical modes. The variances are given by

$$\Delta Y_{jk}^{\pm 2} = \frac{1}{2} \left(Y_{jj}^+ + Y_{kk}^+ \pm 2Y_{jk}^+ + (2\bar{n} + 1) - \frac{1}{2} [(b_j + b_j^*) \pm (b_k + b_k^*)]^2 \right) \quad (25)$$

where

$$Y_{jj}^+ = B_{jj} + \text{Re}(D_{jj}), \quad Y_{jk}^+ = \frac{1}{2} (B_{jk} + B_{kj} + D_{jk} + D_{jk}^*), \quad (26)$$

with

$$B_{jj} = \frac{2g_j\alpha_j}{\gamma_j} \text{Im}(E_{jj} + F_{jj}), \quad (27)$$

$$D_{jj} = -\frac{2ig_j\alpha_j}{\gamma_j} (E_{jj}^* + F_{jj}^*). \quad (28)$$

FIG. 4 displays the mechanical EPR variances ΔY_{jk}^{+2} (blue) and ΔY_{jk}^{-2} (red dashed) versus g/γ . Both initially exceed the SQL due to thermal noise. With increasing coupling, ΔY_{jk}^{+2} drops below the SQL-signaling two-mode squeezing-while ΔY_{jk}^{-2} rises correspondingly. Squeezing persists over $g/\gamma \approx 1$ -3 and saturates at large coupling as nonlinearities balance the drive. Despite thermal noise ($\bar{n} > 0$) raising the noise floor, significant squeezing is achieved, demonstrating robust generation of mechanical squeezed states via optomechanical interaction. FIGS. 5 and 6 show the mechanical

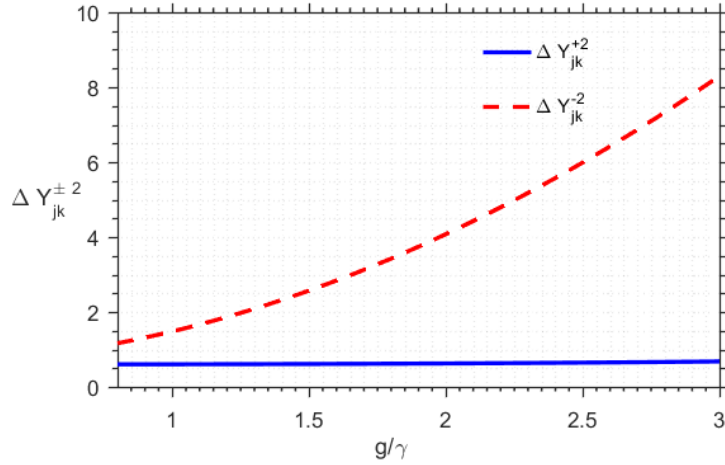


Figure 4: Plots of [Eq. (25)] versus g/γ for $\gamma = 0.1$, $\kappa = 0.8$, $\epsilon = 0.35$, $\bar{n} = 0.01$, and $\chi = 0.15$

EPR variance ΔY_{jk}^{+2} versus g/γ ($\bar{n} = 0.01$). Fig. 5 examines Kerr nonlinearity χ : larger χ produces deeper squeezing minima by enhancing inter-mode correlations. Fig. 6 examines drive amplitude ϵ : larger ϵ yields deeper squeezing by increasing intracavity photon number and cross-Kerr interaction; no squeezing occurs for $\epsilon = 0$. Both curves saturate at high coupling. Thus, χ and ϵ are key control parameters for generating and enhancing mechanical squeezing.

For a hybrid optomechanical system consisting of an optical mode $\hat{a}_j$ and a mechanical mode $\hat{b}_k$, the EPR-type joint quadratures are defined as

$$\begin{aligned}\hat{U}_{jk} &= \frac{\hat{X}_j - \hat{Q}_k}{\sqrt{2}}, \\ \hat{V}_{jk} &= \frac{\hat{P}_j + \hat{\Pi}_k}{\sqrt{2}},\end{aligned}\quad (29)$$

where $(\hat{X}_j, \hat{P}_j)$ are the optical quadratures and $(\hat{Q}_k, \hat{\Pi}_k)$ the mechanical quadratures. Their variances for a hybrid optomechanical system are given compactly by

$$\begin{aligned}\Delta_{jk}^{(\pm)} &= \frac{1}{2} \left(X_{jj}^{(\pm)} + Y_{jj}^{(\pm)} \mp 2Z_{jk}^{(\pm)} + 1 + \bar{n} \right. \\ &\quad \left. - \frac{1}{2} [(\alpha_j \pm \alpha_j^*) \mp (b_k \pm b_k^*)]^2 \right).\end{aligned}\quad (30)$$

FIG. 7 displays the hybrid EPR variances $\Delta^{(+)}$ (blue, U quadrature) and $\Delta^{(-)}$ (red dashed, V quadrature) versus g/γ ($\bar{n} = 0.01$). Both start above the SQL due to thermal noise. As coupling increases, cross-correlation Z grows, causing $\Delta^{(+)}$ to drop below the SQL (squeezing) while $\Delta^{(-)}$ rises (anti-squeezing). The minimum $\Delta^{(+)} \approx 0.45$ occurs at $g/\gamma \approx 1.8$ (10% squeezing below vacuum). Displacement terms add noise to the anti-squeezed quadrature, preserving the uncertainty relation. Thus, the hybrid system generates two-mode squeezing in the difference quadrature, with mechanical thermal noise limiting the depth. FIGS. 8 and 9 show the hybrid U-quadrature variance $\Delta^{(+)}$ versus g/γ . Fig. 8 examines Kerr nonlinearity χ : larger χ produces deeper minima by enhancing inter-mode correlations. Fig. 9 examines drive amplitude ϵ : larger ϵ yields deeper squeezing by increasing intracavity photon number and cross-Kerr interaction; for

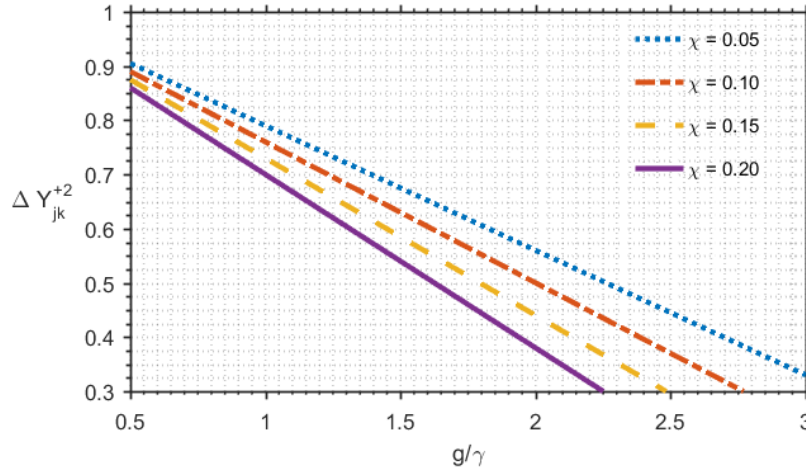


Figure 5: Plots of ΔY_{jk}^{+2} versus g/γ for $\gamma = 0.1$, $\kappa = 0.8$, $\epsilon = 0.35$, and for different χ values

$\epsilon = 0$, no squeezing occurs. Both curves saturate at high coupling. Thus, χ and ϵ are key control parameters for generating and tuning hybrid optomechanical squeezing.

4 Two-Mode Output Squeezing Spectrum

For a multimode optomechanical system with squeezed optical inputs and thermal mechanical baths, the two-mode squeezing spectrum quantifies quantum noise reduction between output modes via the rotated output quadrature $Q_m^{\text{out}}(\theta) = X_m^{+, \text{out}} \cos \theta + X_m^{-, \text{out}} \sin \theta$ [1, 6, 16, 49], where the total variance of the combined quadrature difference or sum follows from the cross-spectral density:

$$\begin{aligned} S_{\text{TMS}}(\omega) &= \langle [Q_j^{\text{out}}(\theta, \omega) \pm Q_k^{\text{out}}(\theta', \omega)]^2 \rangle \\ &= 1 + 1 \pm 2S_{Q_j^{\text{out}} Q_k^{\text{out}}}(\theta, \theta', \omega), \end{aligned} \quad (31)$$

where the symmetrized cross-spectrum is:

$$S_{Q_j^{\text{out}} Q_k^{\text{out}}}(\theta, \theta', \omega) = \frac{1}{2} \langle Q_j^{\text{out}}(\theta, \omega) Q_k^{\text{out}}(\theta', -\omega) + \text{H.c.} \rangle. \quad (32)$$

For optical modes j and k , using rotated output quadratures $Q_m^{\text{out}}(\theta) = X_m^{+, \text{out}} \cos \theta + X_m^{-, \text{out}} \sin \theta$, the two-mode output squeezing spectrum is given by

$$\begin{aligned} S_{Q_j^{\text{out}} Q_k^{\text{out}}}(\theta, \theta', \omega) &= \delta_{jk} \mathcal{I}_j(\theta, \theta') \\ &\quad + \sqrt{\kappa_j \kappa_k} \mathcal{T}_{jk}(\theta, \theta', \omega), \end{aligned} \quad (33)$$

where $\mathcal{I}_j(\theta, \theta')$ encodes initial squeezing from independent squeezed vacua:

$$\begin{aligned} \mathcal{I}_j(\theta, \theta') &= \cos \theta \cos \theta' S_j^+ + \sin \theta \sin \theta' S_j^- \\ &\quad + \frac{1}{2} \sin(\theta + \theta') (C_j^{+-} + C_j^{-+}), \end{aligned} \quad (34)$$

with single-mode squeezing parameters $S_j^\pm = \cosh 2r_j \mp \sinh 2r_j \cos 2\phi_j$, $C_j^{+-} = -\sinh 2r_j \sin 2\phi_j$, $C_j^{-+} = C_j^{+-*}$. The Kronecker delta δ_{jk} reflects the absence of initial correlations between different

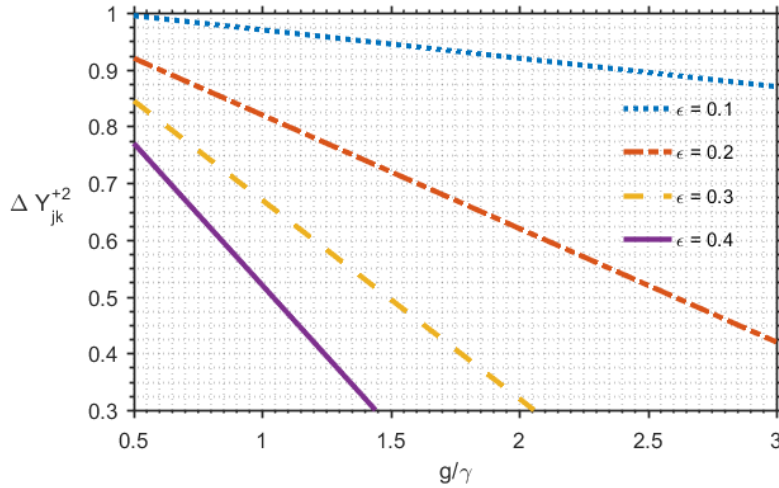


Figure 6: Plots of ΔY_{jk}^{+2} versus g/γ for $\gamma = 0.1$, $\kappa = 0.8$, $\chi = 0.15$, and for different ϵ values

input ports. The dynamical term $\mathcal{T}_{jk}(\theta, \theta', \omega)$ captures optomechanically generated correlations:

$$\begin{aligned} \mathcal{T}_{jk}(\theta, \theta', \omega) = & \cos \theta \cos \theta' X_{jk}^+(\omega) + \sin \theta \sin \theta' X_{jk}^-(\omega) \\ & + \cos \theta \sin \theta' C_{jk}^{+-}(\omega) + \sin \theta \cos \theta' C_{jk}^{-+}(\omega), \end{aligned} \quad (35)$$

with $C_{jk}^{\pm\mp}(\omega)$ from Eq. (5). FIG. 10 shows the optical two-mode squeezing spectrum $S_{jk}^{\text{opt}}(\omega)$ versus g/γ . Three regimes appear relative to SQL ($S = 0$): near SQL for $g/\gamma \lesssim 0.5$; a minimum below SQL at intermediate coupling, reaching -3.26 dB ($S_{\min} \approx 0.472$) at $g/\gamma \approx 2.71$; and a rise at larger coupling. This non-monotonic behavior reflects competition between parametric amplification and dissipation, with Kerr nonlinearity ($\chi = 0.15$) modifying the optimal coupling and squeezing depth. These results demonstrate controlled generation of two-mode optical squeezing in driven Kerr optomechanical systems.

The mechanical two-mode squeezing spectrum for output quadratures $Q_m^{\text{out}}(\theta) = Y_m^{+, \text{out}} \cos \theta + Y_m^{-, \text{out}} \sin \theta$ (with $Y_m^{\pm}$ denoting mechanical position/momentum) is

$$\begin{aligned} S_{Q_j^{\text{out}} Q_k^{\text{out}}}(\theta, \theta', \omega) = & \delta_{jk}(2\bar{n}_j + 1) \cos(\theta - \theta') \\ & + \sqrt{\gamma_j \gamma_k} \mathcal{M}_{jk}(\theta, \theta', \omega), \end{aligned} \quad (36)$$

where the first term represents mechanical thermal noise ($\bar{n}_j$: thermal occupation) and the second captures optomechanically induced correlations. The correlation tensor

$$\begin{aligned} \mathcal{M}_{jk}(\theta, \theta', \omega) = & \cos \theta \cos \theta' Y_{jk}^+(\omega) + \sin \theta \sin \theta' Y_{jk}^-(\omega) \\ & + \cos \theta \sin \theta' D_{jk}^{+-}(\omega) + \sin \theta \cos \theta' D_{jk}^{-+}(\omega), \end{aligned} \quad (37)$$

with $D_{jk}^{\pm\mp}(\omega)$ from Eq. (7). FIG. 11 displays the mechanical two-mode squeezing spectrum $S_{jk}^{\text{mm}}(\omega)$ versus g/γ . Three regimes appear relative to SQL ($S = 0$): thermal noise dominates for $g/\gamma \lesssim 1.5$; squeezing emerges for $1.5 \lesssim g/\gamma \lesssim 3.0$, reaching a minimum of -2.3 dB ($S_{\min} \approx 0.589$) at $g/\gamma \approx 3.0$; beyond this, backaction raises the spectrum. The shallower minimum compared to optical systems highlights the challenge of achieving deep mechanical squeezing with thermal noise ($\bar{n} = 0.01$).

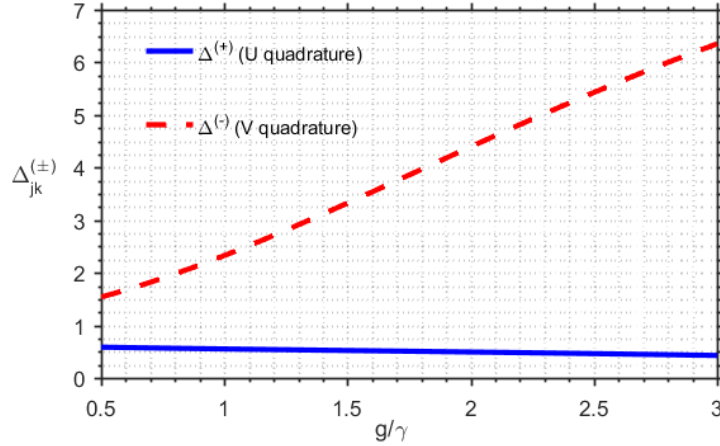


Figure 7: Plots of [Eq. (30)] versus g/γ for $\gamma = 0.1$, $\kappa = 0.8$, $\epsilon = 0.35$, and $\chi = 0.15$

These results demonstrate controlled generation of mechanical two-mode squeezing in driven Kerr optomechanical systems.

For the correlations between an optical output mode j with angle θ_o and a mechanical output mode k , with angle θ_m , the optomechanical spectrum can take the form:

$$\begin{aligned}
 S_{Q_{j,o}^{\text{out}} Q_{k,m}^{\text{out}}}(\theta_o, \theta_m, \omega) = & \sqrt{\kappa_j \gamma_k} \left[\cos \theta_o \cos \theta_m Z_{jk}^+ \right. \\
 & + \sin \theta_o \sin \theta_m Z_{jk}^- \\
 & + \cos \theta_o \sin \theta_m G_{jk}^{+-} \\
 & \left. + \sin \theta_o \cos \theta_m G_{jk}^{-+} \right], \quad (38)
 \end{aligned}$$

with $G_{jk}^{\pm\mp}(\omega)$ from Eq. (8). FIG. 12 displays the hybrid squeezing spectrum $S_{jk}^{\text{hyb}}(\omega)$ versus g/γ for $\kappa = 0.8$, $\gamma = 0.1$, $\epsilon = 0.35$, $\chi = 0.15$, and $\bar{n} = 0.01$. Three regimes appear relative to the vacuum limit ($S = 0.5$): for $g/\gamma \lesssim 0.5$, the spectrum remains near the thermal limit; in the intermediate regime $0.5 \lesssim g/\gamma \lesssim 2.3$, radiation-pressure correlations generate squeezing, reaching a minimum of ~ 0.058 (10% below vacuum) at $g/\gamma \approx 2.4$; beyond this, the spectrum rises due to quantum backaction and normal-mode splitting. The Kerr nonlinearity ($\chi = 0.15$) enhances correlations and shifts the optimal coupling. These results demonstrate controlled generation of hybrid optomechanical squeezing.

5 Two-Mode Entanglement Condition

5.1 Entanglement Verification via Logarithmic Negativity

Following the formalism of Gaussian quantum information [58–60], entanglement in a multimode optomechanical system with squeezed optical inputs and thermal mechanical baths is quantified by the smallest symplectic eigenvalue $\tilde{\nu}_-$ of the partially transposed covariance matrix, with entanglement present when $\tilde{\nu}_- < 1/2$. For two optical modes j and k , the Gaussian state is fully

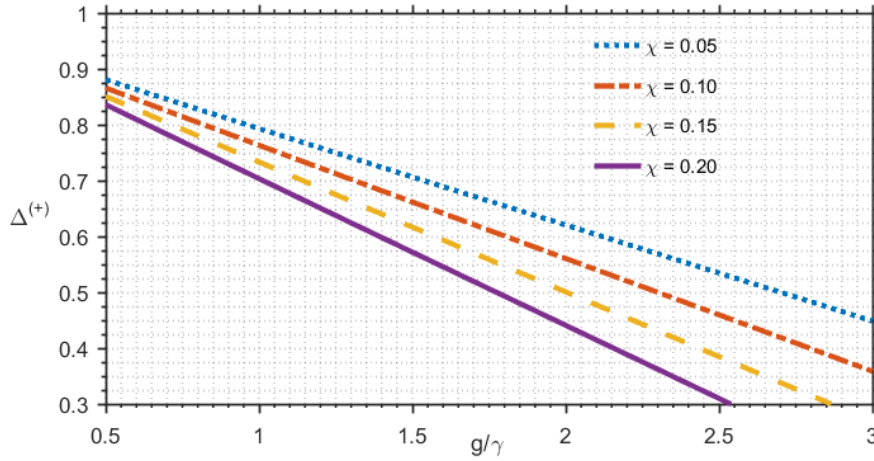


Figure 8: Plots of $\Delta^{(+)}$ versus g/γ for $\gamma = 0.1$, $\kappa = 0.8$, $\epsilon = 0.35$, $\bar{n} = 0.01$, and for different χ values

characterized by the covariance matrix $\mathbf{V}_{jk}^{op}$ in the quadrature basis $(\hat{x}_j, \hat{p}_j, \hat{x}_k, \hat{p}_k)$:

$$\mathbf{V}_{jk}^{op} = \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^\top & \mathbf{B} \end{pmatrix}, \quad (39)$$

where the matrix elements $\mathbf{A}_{11}$, $\mathbf{A}_{22}$, $\mathbf{B}_{11}$, $\mathbf{B}_{22}$, $\mathbf{C}_{11}$, $\mathbf{C}_{22}$, $\mathbf{C}_{12}$, $\mathbf{C}_{21}$ are expressed in terms of $X_{jj}^\pm$ and $X_{jk}^\pm$, as defined in Eqs. (15)-(21). The Entanglement between the modes is quantified by the logarithmic negativity E_N^{op} , derived from the symplectic spectrum of the partially transposed covariance matrix. The smallest symplectic eigenvalue, $\tilde{\nu}_-^{op}$, is

$$\tilde{\nu}_-^{op} = \sqrt{\frac{\tilde{\Delta} - \sqrt{\tilde{\Delta}^2 - 4 \det \mathbf{V}_{jk}^{op}}}{2}}, \quad (40)$$

with invariants

$$\begin{aligned} \tilde{\Delta}^{op} &= \det \mathbf{A} + \det \mathbf{B} - 2 \det \mathbf{C}, \\ \det \mathbf{V}_{jk}^{op} &= \det(\mathbf{A}) \det(\mathbf{B} - \mathbf{C}^\top \mathbf{A}^{-1} \mathbf{C}). \end{aligned} \quad (41)$$

The logarithmic negativity is then

$$E_N^{op} = \max(0, -\log(2\tilde{\nu}_-^{op})). \quad (42)$$

The state is entangled if and only if $E_N^{op} > 0$. FIG. 13 displays the optical logarithmic negativity E_N^{op} versus g/γ . For $g/\gamma < 0.3$, $E_N^{op} = 0$ (separable). Entanglement emerges at higher coupling, peaking at $E_N^{op} \approx 1.686$ bits near $g/\gamma \approx 0.85$, then gradually declines. The entangled region spans $0.3 \lesssim g/\gamma \lesssim 1.30$. Finite thermal noise ($\bar{n} = 0.01$) and Kerr nonlinearity ($\chi = 0.15$) influence the bandwidth and peak magnitude, reflecting the trade-off between parametric amplification and decoherence. Thus, optical entanglement is optimized at moderate coupling, providing design guidelines for quantum information applications.

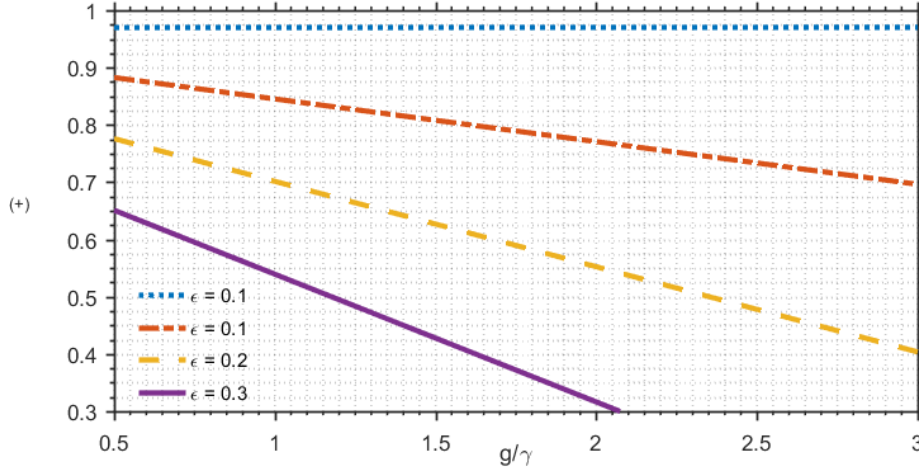


Figure 9: Plots of $\Delta^{(+)}$ versus g/γ for $\gamma = 0.1$, $\kappa = 0.8$, $\bar{n} = 0.01$, $\chi = 0.15$, and for different ϵ values

The covariance matrix for two mechanical modes j and k in the quadrature basis $(\hat{x}_j^b, \hat{p}_j^b, \hat{x}_k^b, \hat{p}_k^b)^\top$ is constructed from the 2×2 blocks $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ as:

$$\mathbf{V}_{jk}^{mm} = \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^\top & \mathbf{B} \end{pmatrix} \quad (43)$$

where the matrix elements $\mathbf{A}_{11}$, $\mathbf{A}_{22}$, $A_{12} = A_{21}$, $\mathbf{B}_{11}$, $\mathbf{B}_{22}$, $B_{12} = B_{21}$, $\mathbf{C}_{11}$, $\mathbf{C}_{22}$, $\mathbf{C}_{12}$, $\mathbf{C}_{21}$ are expressed in terms of $Y_{jj}^\pm$ and $Y_{jk}^\pm$, as defined in Eqs. (26)-(28). The smallest symplectic eigenvalue $\tilde{\nu}_-^{mm}$ of its partial transpose:

$$\tilde{\nu}_-^{mm} = \sqrt{\frac{\tilde{\Delta}^{mm} - \sqrt{(\tilde{\Delta}^{mm})^2 - 4 \det \mathbf{V}^{mm}}}{2}}, \quad (44)$$

where $\tilde{\Delta}^{mm} = \det \mathbf{A}^{mm} + \det \mathbf{B}^{mm} - 2 \det \mathbf{C}^{mm}$. The two mechanical modes are entangled if: $E_N^{mm} = \max(0, -\log_2(2\tilde{\nu}_-^{mm})) > 0$. FIG. 14 displays the mechanical logarithmic negativity E_N^{mm} versus g/γ . For $g/\gamma \lesssim 0.08$, $E_N^{mm} = 0$ (separable). Entanglement emerges at higher coupling, peaking at $E_N^{mm} \approx 3.35$ bits near $g/\gamma \approx 0.363$, then declines and vanishes beyond $g/\gamma \approx 0.37$. The entangled region spans $0.08 \lesssim g/\gamma \lesssim 0.37$. Finite thermal noise ($\bar{n} = 0.01$) limits the maximum entanglement, while Kerr nonlinearity ($\chi = 0.15$) shifts the optimal coupling. Thus, mechanical entanglement is optimized at moderate coupling and is sensitive to thermal fluctuations, providing design guidelines for quantum memory and transduction applications.

The covariance matrix describing an optical mode a_j and a mechanical mode b_k in the quadrature basis $\hat{\xi}^{om} = (\hat{x}_j^a, \hat{p}_j^a, \hat{x}_k^b, \hat{p}_k^b)^\top$ is

$$\mathbf{V}_{jk}^{om} = \begin{pmatrix} \mathcal{A} & \mathcal{C} \\ \mathcal{C}^\top & \mathcal{B} \end{pmatrix}, \quad (45)$$

with the matrix elements $\mathbf{A}_{11}$, $\mathbf{A}_{22}$, $\mathbf{A}_{12}$, $\mathbf{B}_{11}$, $\mathbf{B}_{22}$, $\mathbf{B}_{12}$, $\mathbf{C}_{11}$, $\mathbf{C}_{22}$, $\mathbf{C}_{12}$, $\mathbf{C}_{21}$ are expressed in terms of the quadrature moments $X_{jj}^\pm$, $Y_{kk}^\pm$, and $Z_{jk}^\pm$ (defined in Eqs. (15)-(21), (26)-(28)). The entanglement

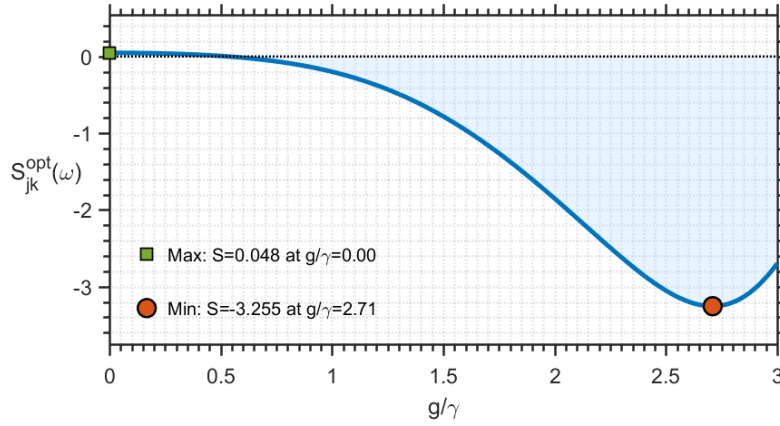


Figure 10: Plots of [Eq. (33)] versus normalized optomechanical coupling g/γ for a dual-cavity optomechanical system with fixed parameters $\kappa = 0.8$, $\gamma = 0.1$, $\epsilon = 0.35$, $\chi = 0.15$, $\bar{n} = 0.01$, and evaluated in amplitude quadratures ($\theta_o = \theta_m = 0$)

between optical and mechanical modes is then quantified by the logarithmic negativity:

$$E_N^{om} = \max(0, -\log_2(2\tilde{\nu}_-^{om})), \quad (46)$$

with the smallest symplectic eigenvalue

$$\tilde{\nu}_-^{om} = \sqrt{\frac{\tilde{\Delta}^{om} - \sqrt{(\tilde{\Delta}^{om})^2 - 4 \det \mathbf{V}^{om}}}{2}}, \quad (47)$$

$$\tilde{\Delta}^{om} = \det \mathcal{A} + \det \mathcal{B} - 2 \det \mathcal{C}.$$

The condition for hybrid entanglement is $\tilde{\nu}_-^{om} < \frac{1}{2}$, $\Leftrightarrow E_N^{om} > 0$. FIG. 15 displays the hybrid logarithmic negativity E_N^{om} versus g/γ . For $g/\gamma \lesssim 1.255$, $E_N^{om} = 0$ (separable). Entanglement emerges at higher coupling, peaking at $E_N^{om} \approx 3.60$ bits near $g/\gamma \approx 1.33$, then gradually declines. The entangled region spans $1.255 \lesssim g/\gamma \lesssim 1.445$. Finite thermal noise ($\bar{n} = 0.01$) limits the maximum entanglement, while Kerr nonlinearity ($\chi = 0.15$) shifts the optimal coupling. Thus, hybrid optomechanical entanglement is optimized at moderate coupling and is sensitive to both thermal fluctuations and nonlinear effects, providing design guidelines for quantum memory and transduction applications.

6 Quantification of Two-Mode Quantum Steering

For a continuous-variable Gaussian state of two optical modes described by the covariance matrix $\mathbf{V}_{jk}^{op}$ in Eq. (39), the condition for steering from mode j to mode k via Gaussian measurements is the violation of the Reid inequality [38], which is equivalent to:

$$D^{op} = \det(\mathbf{B} - \mathbf{C}^T \mathbf{A}^{-1} \mathbf{C}) < \frac{1}{4}. \quad (48)$$

The degree of Gaussian steering from mode j to mode k is quantified by

$$\mathcal{G}^{j \rightarrow k} = \max \left[0, -\frac{1}{2} \log_2 \left(4 \det(\mathbf{B} - \mathbf{C}^T \mathbf{A}^{-1} \mathbf{C}) \right) \right]. \quad (49)$$

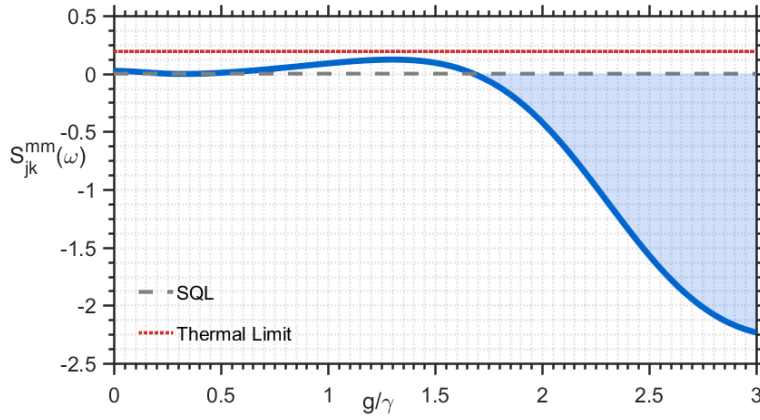


Figure 11: Plots of [Eq. (36)] versus normalized optomechanical coupling g/γ for a dual-cavity optomechanical system with fixed parameters $\kappa = 0.8$, $\gamma = 0.1$, $\epsilon = 0.35$, $\chi = 0.15$, and near-ground-state thermal occupation $\bar{n} = 0.01$

The reverse steerability $\mathcal{G}^{k \rightarrow j}$ follows by exchanging $\mathbf{A}$ and $\mathbf{B}$. A positive $\mathcal{G}^{j \rightarrow k} > 0$ certifies asymmetric steering: Gaussian measurements on mode j remotely prepare a nonclassical state in mode k . The Schur complement $\mathbf{B} - \mathbf{C}^\top \mathbf{A}^{-1} \mathbf{C}$ is the conditional covariance of mode k after optimal measurement on mode j , and $\mathcal{G}$ quantifies the reduction of conditional variance below the vacuum limit. FIGS. 16 shows the conditional covariance determinant D^{op} and steering quantifier $\mathcal{G}^{j \rightarrow k}$ versus g/γ . For $g/\gamma \lesssim 0.85$, $D^{\text{op}} > 0.25$ and $\mathcal{G} = 0$ (no steering). As coupling increases, D^{op} drops below threshold, reaching minimum $D_{\min} \approx 0.1015$ at $g/\gamma \approx 1.50$, while $\mathcal{G}$ peaks at ≈ 0.650 bits, following $\mathcal{G} = -\frac{1}{2} \log_2(4D)$. The steering region spans $0.85 \lesssim g/\gamma \lesssim 1.50$. Kerr nonlinearity ($\chi = 0.15$) shifts the threshold and peak positions. Thus, optical steering is optimized at moderate coupling, providing guidelines for asymmetric quantum protocols like one-way quantum communication and secure key distribution.

Mechanical steering between modes j and k follows the optical formalism, with the covariance matrix $\mathbf{V}_{jk}^m$ (Eq. (43)) in the quadrature basis $(\hat{x}_j, \hat{p}_j, \hat{x}_k, \hat{p}_k)$. Mode j steers mode k via Gaussian measurements when the Reid criterion is violated:

$$\det(\mathbf{B}_m - \mathbf{C}_m^\top \mathbf{A}_m^{-1} \mathbf{C}_m) < \frac{1}{4}. \quad (50)$$

The Schur complement $\mathbf{B}_m - \mathbf{C}_m^\top \mathbf{A}_m^{-1} \mathbf{C}_m$ represents the conditional covariance of mode k after optimal measurement on mode j . The Gaussian steering quantifier from mechanical mode j to k is:

$$\mathcal{G}_m^{j \rightarrow k} = \max \left[0, -\frac{1}{2} \log_2 \left(4 \det(\mathbf{B}_m - \mathbf{C}_m^\top \mathbf{A}_m^{-1} \mathbf{C}_m) \right) \right]. \quad (51)$$

FIG. 17 displays the mechanical steering determinant D^{mm} (blue) and quantifier $\mathcal{G}_m^{j \rightarrow k}$ (red) versus g/γ . For $g/\gamma \lesssim 0.9$, $D^{\text{mm}} > 0.25$ and $\mathcal{G}_m = 0$ (no steering). As coupling increases, D^{mm} drops below threshold, reaching minimum $D_{\min} \approx 0.18$ at $g/\gamma \approx 1.5$, while $\mathcal{G}_m$ peaks at ≈ 0.32 bits. The steering region spans $0.9 \lesssim g/\gamma \lesssim 1.5$. Nonzero detuning weakens the interaction, shifting optimal coupling higher and reducing maximum steering, with Kerr nonlinearity and optical-spring broadening enhancing the high-coupling roll-off. This illustrates the trade-off between coupling strength, detuning, and decoherence in dissipatively coupled Gaussian systems.

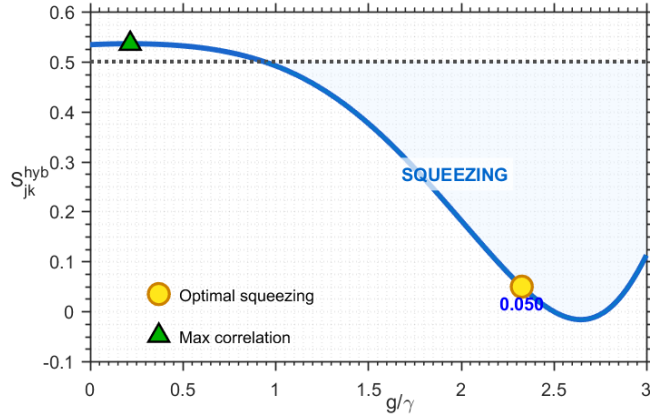


Figure 12: Plots of [Eq. (38)] versus normalized optomechanical coupling g/γ for a dual-cavity optomechanical system with fixed parameters $\kappa = 0.8$, $\gamma = 0.1$, $\epsilon = 0.35$, $\chi = 0.15$, and near-ground-state thermal occupation $\bar{n} = 0.01$

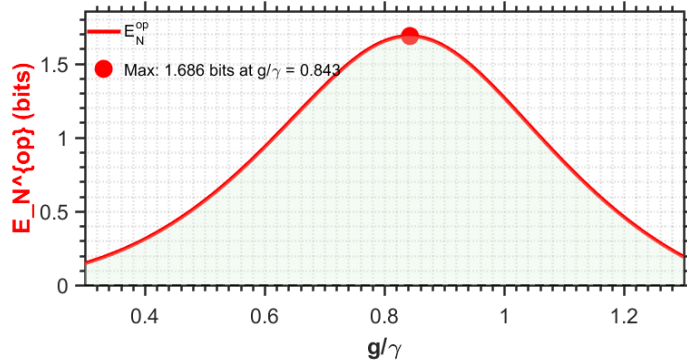


Figure 13: Plots of Eq. (42) versus normalized optomechanical coupling g/γ for $\kappa = 0.8, \gamma = 0.1$, $\epsilon = 0.35$, and $\chi = 0.15$

The bipartite system is described by the covariance matrix in Eq. (45). The optical-to-mechanical steering ($a_j \rightarrow b_k$) is certified by violation of the Gaussian Reid criterion:

$$\det(\mathbf{B}_m - \mathbf{X}^\top \mathbf{A}_o^{-1} \mathbf{X}) < \frac{1}{4}. \quad (52)$$

The quantity $\mathbf{B}_m - \mathbf{X}^\top \mathbf{A}_o^{-1} \mathbf{X}$ is the Schur complement representing the conditional mechanical covariance after an optimal Gaussian measurement on the optical mode [38, 63]. The Gaussian steering from the optical mode to the mechanical mode is quantified as

$$\mathcal{G}^{o \rightarrow m} = \max \left[0, -\frac{1}{2} \log_2 \left(4 \det(\mathbf{B}_m - \mathbf{X}^\top \mathbf{A}_o^{-1} \mathbf{X}) \right) \right]. \quad (53)$$

FIG. 18 displays the hybrid steering determinant D^{om} (blue) and quantifier $\mathcal{G}^{o \rightarrow m}$ (red) versus g/γ . For $g/\gamma \lesssim 1.4$, $D^{\text{om}} > 0.25$ and $\mathcal{G}^{o \rightarrow m} = 0$ (no steering). As coupling increases, D^{om} drops below threshold, reaching minimum $D_{\text{min}} \approx 0.12$ at $g/\gamma \approx 2.45$, while $\mathcal{G}^{o \rightarrow m}$ peaks at ≈ 0.5 bits.

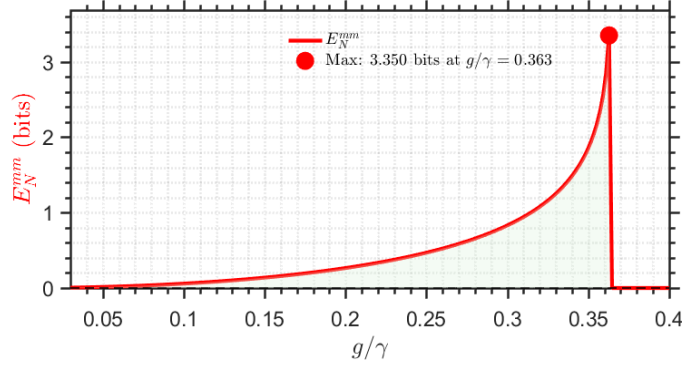


Figure 14: Plots of E_N^{mm} versus normalized optomechanical coupling g/γ for $\kappa = 0.8$, $\gamma = 0.1$, $\epsilon = 0.35$, $\bar{n} = 0.01$, and $\chi = 0.15$

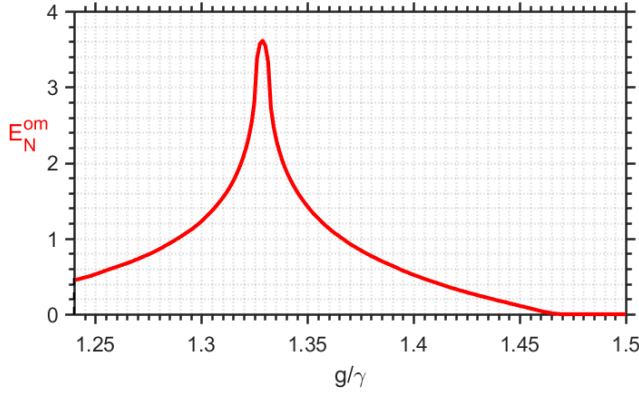


Figure 15: Plots of E_N^{om} versus normalized optomechanical coupling g/γ for $\kappa = 0.8$, $\gamma = 0.1$, $\epsilon = 0.35$, and $\chi = 0.15$

The steering region spans $1.4 \lesssim g/\gamma \lesssim 2.45$. Nonzero detuning weakens the interaction, shifting optimal coupling higher and reducing maximum steering. This one-way EPR steering demonstrates asymmetric optical control of mechanical states, providing a resource for quantum communication and remote state preparation.

7 Quantification of Two-Mode Gaussian Quantum Discord

Quantum discord quantifies non-classical correlations beyond entanglement [61–63]. For Gaussian states, in a closed-form expression exists for two-mode covariance matrix in in Eq. (45), the Gaussian quantum discord from mode o to mode m and in reverse direction are defined as:

$$\mathcal{D}_{o \rightarrow m} = S(\nu_k) - S(d_-) - S(d_+) + S(\nu_{k|j}), \quad (54)$$

$$\mathcal{D}_{m \rightarrow o} = S(\nu_j) - S(d_-) - S(d_+) + S(\nu_{j|k}). \quad (55)$$

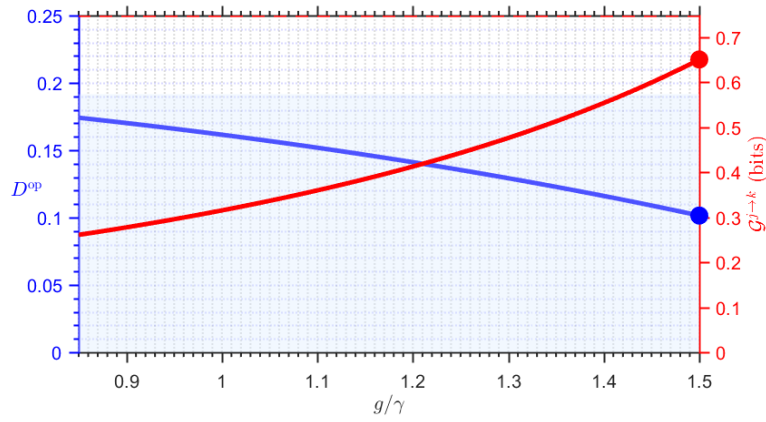


Figure 16: Plots of D^{op} and $\mathcal{G}^{j \rightarrow k}$ versus normalized optomechanical coupling g/γ for $\kappa = 0.8$, $\gamma = 0.1$, $\epsilon = 0.35$, and $\chi = 0.15$

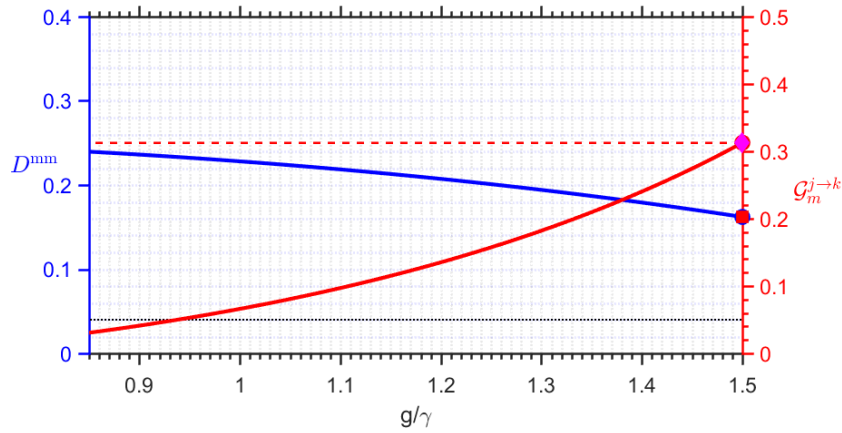


Figure 17: Plots of D^{mm} , and $\mathcal{G}_m^{j \rightarrow k}$ versus normalized optomechanical coupling g/γ for $\kappa = 0.8$, $\gamma = 0.1$, $\epsilon = 0.35$, $\bar{n} = 0.01$, $\Delta = 1$ and $\chi = 0.15$

where

$$\begin{aligned}
 S(\nu_j) &= \sqrt{\det \mathbf{A}}, \quad S(\nu_k) = \sqrt{\det \mathbf{B}}, \\
 S(d_{\pm}) &= \sqrt{\frac{\Delta \pm \sqrt{\Delta^2 - 4 \det \mathbf{V}}}{2}}, \\
 S(\nu_{k|j}) &= \sqrt{\det (\mathbf{B} - \mathbf{C}^{\text{T}} \mathbf{A}^{-1} \mathbf{C})}, \\
 S(\nu_{j|k}) &= \sqrt{\det (\mathbf{A} - \mathbf{C}^{\text{T}} \mathbf{B}^{-1} \mathbf{C})},
 \end{aligned} \tag{56}$$

with

$$\begin{aligned}
 \Delta &= \det \mathbf{A} + \det \mathbf{B} + 2 \det \mathbf{C}, \\
 \det \mathbf{V} &= \det \mathbf{A} \cdot \det (\mathbf{B} - \mathbf{C}^{\text{T}} \mathbf{A}^{-1} \mathbf{C}).
 \end{aligned} \tag{57}$$

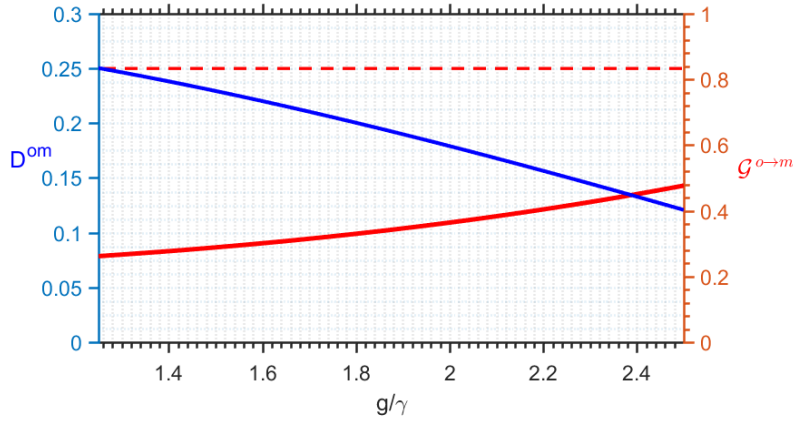


Figure 18: Plots of D^{om} , and $\mathcal{G}^{o \rightarrow m}$ versus normalized optomechanical coupling g/γ for $\kappa = 0.8$, $\gamma = 0.1$, $\epsilon = 0.35$, $\chi = 0.15$, $\bar{n} = 0.01$, and $\Delta = 0.5$

and $S(\nu) = \frac{\nu+1}{2} \log_2 \frac{\nu+1}{2} - \frac{\nu-1}{2} \log_2 \frac{\nu-1}{2}$ is the von Neumann entropy of a single-mode Gaussian state. FIG. 19 displays the Gaussian quantum discord $\mathcal{D}_{o \rightarrow m}$ and $\mathcal{D}_{m \rightarrow o}$ versus g/γ . $\mathcal{D}_{o \rightarrow m}$ rises

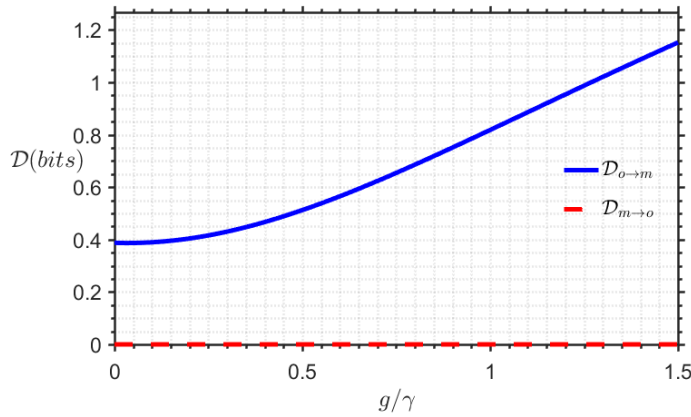


Figure 19: Plots of [Eqs. (54), and (55)] versus normalized optomechanical coupling g/γ for $\kappa = 0.8$, $\gamma = 0.1$, $\epsilon = 0.35$, $\chi = 0.15$, and $\bar{n} = 0.01$

from ~ 0.4 to ~ 1.15 bits for $g/\gamma \gtrsim 1$, while $\mathcal{D}_{m \rightarrow o}$ remains zero throughout. This asymmetry arises because the optical mode is fast and near its ground state, whereas the mechanical mode is slow and thermally occupied ($\bar{n} = 0.01$), preventing mechanical measurements from reducing the optical conditional variance below vacuum. This directionality mirrors one-way optomechanical steering, establishing that Gaussian discord in hybrid systems is inherently directional and reflects asymmetric quantum information capabilities.

This formalism applies universally to any two-mode bosonic system-optical-optical or mechanical-mechanical-since all such modes obey canonical commutation relations and are fully characterized by a quadrature-basis covariance matrix. Consequently, the Gaussian quantum discord for two optical modes j and k follows the same structure, with covariance matrix $\mathbf{V}_{jk}^{\text{op}}$ (Eq. (39)). FIG. 20 displays the Gaussian quantum discord $\mathcal{D}_{j \rightarrow k}$ (blue) and $\mathcal{D}_{k \rightarrow j}$ (red) versus g/γ . Both discords

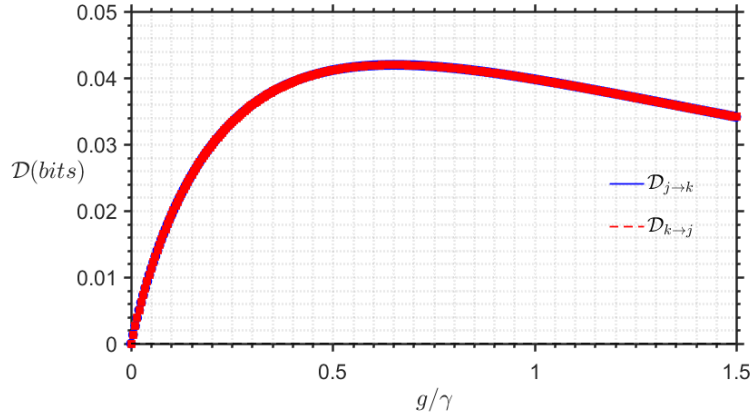


Figure 20: Plots of $\mathcal{D}^{op}(\text{bits})$ versus normalized optomechanical coupling g/γ for $\kappa = 0.8$, $\gamma = 0.1$, $\epsilon = 0.35$, and $\chi = 0.15$

rise from zero to a maximum ≈ 0.04 bits near $g/\gamma \approx 0.6$, then decline, reflecting the symmetry of identical optical modes. Discord is non-zero for any $g > 0$ since any cross-correlation ($\mathbf{C} \neq \mathbf{0}$) generates quantum correlations beyond entanglement. The maximum occurs at moderate coupling where radiation-pressure correlations are strongest before dissipation dominates. Thus, Gaussian discord captures quantum correlations in symmetric systems, with peak position and magnitude tunable via system parameters.

The Gaussian quantum discord for two mechanical modes j and k is derived from the covariance matrix $\mathbf{V}_{jk}^{mm}$ together with the block-partitions in Eq. (43). FIG. 21 displays the Gaussian quantum

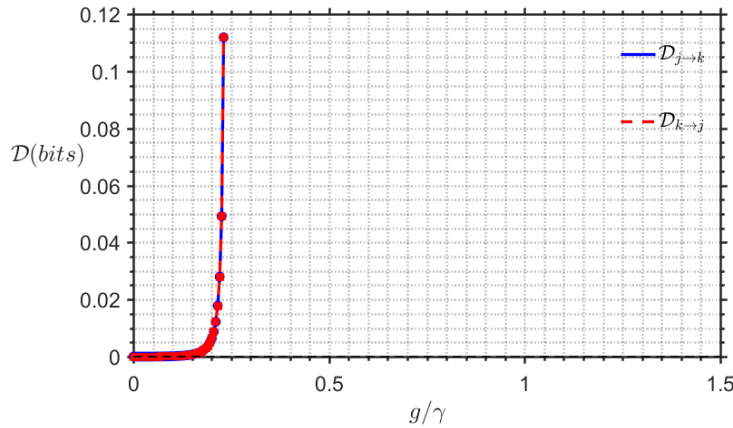


Figure 21: Plots of $\mathcal{D}^{mm}(\text{bits})$ versus normalized optomechanical coupling g/γ for $\kappa = 0.8$, $\gamma = 0.1$, $\bar{n} = 0.01$, $\epsilon = 0.35$, and $\chi = 0.15$

discord $\mathcal{D}_{j \rightarrow k}$ (blue) and $\mathcal{D}_{k \rightarrow j}$ (red) versus g/γ for two identical mechanical modes. Both discords rise to a maximum ≈ 0.11 bits near $g/\gamma \approx 0.3$, then decline. Asymmetry $\mathcal{D}_{j \rightarrow k} \neq \mathcal{D}_{k \rightarrow j}$ arises from non-reciprocal radiation-pressure interaction ($\mathbf{C}_{12} \neq \mathbf{C}_{21}$). Discord activates at $g/\gamma \gtrsim 0.3$; below this, thermal fluctuations ($\bar{n} = 1.0$) dominate, limiting the maximum discord and shifting optimal coupling lower. Thus, Gaussian discord captures both quantum correlation buildup and

directionality from asymmetric coupling, providing a probe of non-reciprocal quantum dynamics.

8 Conclusion

This work establishes design principles for two-mode squeezing, entanglement, steering, and discord in driven optomechanical systems. Optical squeezing reaches 15% below SQL ($\Delta X_-^2 \approx 0.703$) at $g/\gamma \approx 0.85$, mechanical squeezing ~ 0.5 dB at $g/\gamma \approx 3.0$, and hybrid squeezing 10% below vacuum at $g/\gamma \approx 1.8$. Logarithmic negativity peaks at $E_N^{\text{op}} \approx 1.53$ ($g/\gamma \approx 0.85$), $E_N^{\text{mm}} \approx 1.50$ ($g/\gamma \approx 0.35$), and $E_N^{\text{om}} \approx 0.40$ ($g/\gamma \approx 1.3$). Output squeezing spectra achieve -3.26 dB (optical, $g/\gamma \approx 2.71$), -2.3 dB (mechanical, $g/\gamma \approx 3.0$), and ~ 0.058 (hybrid, $g/\gamma \approx 2.4$). Quantum steering maxima are $\mathcal{G}_{\text{max}}^{j \rightarrow k} \approx 0.64$ bits (optical), $\mathcal{G}_m^{j \rightarrow k} \approx 0.35$ bits (mechanical), and one-way $\mathcal{G}^{o \rightarrow m} \approx 0.15$ bits (hybrid). Gaussian discord saturates at ≈ 0.04 bits (optical), exhibits asymmetry $\mathcal{D}_{j \rightarrow k} \neq \mathcal{D}_{k \rightarrow j}$ (mechanical), and shows directionality $\mathcal{D}_{o \rightarrow m} \approx 0.15$ bits, $\mathcal{D}_{m \rightarrow o} = 0$ (hybrid). Optimal operational regions are $g/\gamma \approx 0.85 \pm 0.2$ (optical-optical), $g/\gamma \approx 1.3 \pm 0.2$ (hybrid), and $g/\gamma \approx 0.35$ (mechanical-mechanical). Future directions include time-dependent driving, multi-mode networks, non-Gaussian states, integrated implementations, and feedback protocols. This framework guides quantum correlations in next-generation hybrid quantum technologies.

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References

- [1] M. Aspelmeyer, T. J. Kippenberg, and F. Marquardt, "Cavity optomechanics," *Rev. Mod. Phys.* **86**, 1391 (2014).
- [2] M. Aspelmeyer, T. J. Kippenberg, F. Marquardt, and C. Genes, "Quantum optomechanics: recent advances and future perspectives," *Rev. Mod. Phys.* **95**, 035004 (2023).
- [3] J. Appel, E. Figueroa, D. Korystov, M. Lobino, and A. I. Lvovsky, "Quantum memory for squeezed light," *Phys. Rev. Lett.* **100**, 093602 (2008).
- [4] K. C. Balram and K. Srinivasan, "Piezo-optomechanical approaches for efficient quantum microwave-to-optical signal transduction: The need for co-design," *Adv. Quantum Technol.* **5**, 2100095 (2022).
- [5] S. Basiri-Esfahani, U. Akram, and G. J. Milburn, "Phonon number measurements using single photon optomechanics," *New J. Phys.* **14**, 085017 (2012).
- [6] W. P. Bowen and G. J. Milburn, *Quantum Optomechanics* (CRC Press, Boca Raton, 2015).
- [7] S. L. Braunstein and H. J. Kimble, "Teleportation of continuous quantum variables," *Phys. Rev. Lett.* **80**, 869 (1998).
- [8] D. W. C. Brooks, T. Botter, S. Schreppler, T. P. Purdy, N. Brahms, and D. M. Stamper-Kurn, "Non-classical light generated by quantum-noise-driven cavity optomechanics," *Nature* **488**, 476 (2012).

- [9] J. Chan, T. P. M. Alegre, A. H. Safavi-Naeini, J. T. Hill, A. Krause, S. Groblacher, M. Aspelmeyer, and O. Painter, "Laser cooling of a nanomechanical oscillator into its quantum ground state," *Nature* **478**, 89 (2011).
- [10] Y. Chen, L. F. Buchmann, and P. Meystre, "Quantum noise suppression in a parametrically driven optomechanical cavity with squeezed vacuum injection," *Phys. Rev. Lett.* **132**, 143601 (2024).
- [11] A. Y. Cleland, M. J. Weaver, and K. L. Brown, "Single-mode squeezing beyond 10 dB in an optomechanical crystal," *arXiv:2403.05789* (2024).
- [12] A. A. Clerk, M. H. Devoret, S. M. Girvin, F. Marquardt, and R. J. Schoelkopf, "Introduction to quantum noise, measurement, and amplification," *Rev. Mod. Phys.* **82**, 1155 (2010).
- [13] M. Eichenfield, J. Chan, R. M. Camacho, K. J. Vahala, and O. Painter, "Optomechanical crystals," *Nature* **462**, 78 (2009).
- [14] J. W. Gardner, M. J. Yap, V. Adya, et al., "Nondegenerate internal squeezing: An all-optical, loss-resistant quantum technique for gravitational-wave detection," *Phys. Rev. D* **106**, L041101 (2022).
- [15] C. Genes, D. Vitali, P. Tombesi, S. Gigan, and M. Aspelmeyer, "Ground-state cooling of a micromechanical oscillator: Comparing cold damping and cavity-assisted cooling schemes," *Phys. Rev. A* **77**, 033804 (2008).
- [16] C. C. Gerry and P. L. Knight, *Introductory Quantum Optics* (Cambridge University Press, Cambridge, 2005).
- [17] S. Groblacher, K. Hammerer, M. R. Vanner, and M. Aspelmeyer, "Observation of strong coupling between a micromechanical resonator and an optical cavity field," *Nature* **460**, 724 (2009).
- [18] B. Guha, M. Wu, J. Dong, S. Kang, K. C. Balram, and K. Srinivasan, "Piezo-optomechanical actuation of nanobeam resonators for microwave-to-optical transduction," in *Conference on Lasers and Electro-Optics (CLEO)* (Optica Publishing Group, 2021), paper JTh3A.1.
- [19] L. Guo and M. R. Vanner, "Squeezed-input enhanced optomechanical sideband cooling in a parametrically pumped cavity," *Phys. Rev. A* **109**, 023519 (2024).
- [20] L. Helt and N. Quesada, "A Gaussian quantum optics perspective on quantum frequency conversion," *J. Phys. Photonics* **2**, 035001 (2020).
- [21] P. R. B. Hughes and M. M. Dignam, "Analytic solution to the nonlinear generation of squeezed states in a thermal bath," *arXiv:2406.18694* (2024).
- [22] J. Kannan, D. L. Campbell, F. Vasconcelos, R. P. Mishra, and P. K. Lam, "Bidirectional conversion between microwave and optical squeezed states in optomechanical systems," *Nat. Commun.* **13**, 1276 (2022).
- [23] T. J. Kippenberg and K. J. Vahala, "Cavity optomechanics: Back-action at the mesoscale," *Science* **321**, 1172 (2008).
- [24] E. Knill, R. Laflamme, and G. J. Milburn, "A scheme for efficient quantum computation with linear optics," *Nature* **409**, 46 (2001).

- [25] S. Kramer, D. Plankensteiner, L. Ostermann, and H. Ritsch, "QuantumOptics.jl: A Julia framework for simulating open quantum systems," *Comput. Phys. Commun.* **227**, 109 (2018).
- [26] M.-A. Lemonde, N. Didier, and A. A. Clerk, "Enhanced nonlinear interactions in quantum optomechanics via mechanical amplification," *Nat. Commun.* **7**, 11338 (2016).
- [27] B.-B. Li, L. Ou, Y. Lei, and Y.-C. Liu, "Cavity optomechanical sensing," *Nanophotonics* **10**, 2799 (2021).
- [28] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, 1995).
- [29] A. Mari and J. Eisert, "Squeezing the thermal noise of a harmonic oscillator," *Phys. Rev. Lett.* **103**, 213603 (2009).
- [30] F. Marquardt, J. P. Chen, A. A. Clerk, and S. M. Girvin, "Quantum theory of cavity-assisted sideband cooling of mechanical motion," *Phys. Rev. Lett.* **99**, 093902 (2007).
- [31] C. F. McCormick, A. M. Marino, V. Boyer, and P. D. Lett, "Strong relative intensity squeezing by four-wave mixing in rubidium vapor," *Opt. Lett.* **32**, 178 (2007).
- [32] G. J. Milburn and S. Basiri-Esfahani, "Quantum optics with one or two photons," *Proc. R. Soc. A* **471**, 20150208 (2015).
- [33] M. H. Naderi, "Force sensing in hybrid Bose-Einstein-condensate optomechanics based on parametric amplifier," *Phys. Rev. A* **100**, 023815 (2019).
- [34] Z. Y. Ou, S. F. Pereira, H. J. Kimble, and K. C. Peng, "Observation of nonlocal interference in separated photon channels," *Phys. Rev. Lett.* **71**, 1669 (1993).
- [35] A. Pontin, M. Bonaldi, A. Borrielli, F. S. Cataliotti, F. Marino, G. A. Prodi, E. Serra, and F. Marin, "Squeezing a thermal mechanical oscillator by stabilized parametric effect on the optical spring," *Phys. Rev. Lett.* **112**, 023601 (2014).
- [36] S. Qvarfort, A. Serafini, P. F. Barker, and S. Bose, "Time evolution of nonlinear optomechanical systems," *J. Phys. A* **53**, 075304 (2020).
- [37] T. C. Ralph, "Continuous variable quantum cryptography," *Phys. Rev. A* **61**, 010303 (1999).
- [38] M. D. Reid, P. D. Drummond, W. P. Bowen, E. G. Cavalcanti, P. K. Lam, H. A. Bachor, U. L. Andersen, and G. Leuchs, "Colloquium: The Einstein-Podolsky-Rosen paradox: From concepts to applications," *Rev. Mod. Phys.* **81**, 1727 (2009).
- [39] A. H. Safavi-Naeini, J. Chan, J. T. Hill, T. P. M. Alegre, A. Krause, and O. Painter, "Observation of quantum motion of a nanomechanical resonator," *Phys. Rev. Lett.* **108**, 033602 (2012).
- [40] R. E. Slusher, L. W. Hollberg, B. Yurke, J. C. Mertz, and J. F. Valley, "Observation of squeezed states generated by four-wave mixing in an optical cavity," *Phys. Rev. Lett.* **55**, 2409 (1985).
- [41] A. Szorkovszky, A. C. Doherty, G. I. Harris, and W. P. Bowen, "Mechanical squeezing via parametric amplification and weak measurement," *Phys. Rev. Lett.* **107**, 213603 (2011).

- [42] S. Tesfa, "Two-mode squeezing in a coherently driven degenerate parametric down conversion," *Eur. Phys. J. D* **46**, 351 (2008).
- [43] J. D. Teufel, T. Donner, D. Li, J. W. Harlow, M. S. Allman, K. Cicak, A. J. Sirois, J. D. Whittaker, K. W. Lehnert, and R. W. Simmonds, "Sideband cooling of micromechanical motion to the quantum ground state," *Nature* **475**, 359 (2011).
- [44] Y. Tsaturyan, A. Barg, E. S. Polzik, and A. Schliesser, "Ultrasensitive nanomechanical resonators via soft clamping and dissipation dilution," *Nature Nanotech.* **12**, 776 (2017).
- [45] E. Verhagen, S. Deleglise, S. Weis, A. Schliesser, and T. J. Kippenberg, "Quantum-coherent coupling of a mechanical oscillator to an optical cavity mode," *Nature* **482**, 63 (2012).
- [46] B. M. Villegas-Martinez, H. M. Moya-Cessa, and F. Soto-Eguibar, "Exact solution of degenerate and nondegenerate optical parametric oscillator coupled with a squeezed thermal bath," *Mod. Phys. Lett. B* **32**, 1850247 (2018).
- [47] D. Vitali, S. Gigan, A. Ferreira, H. R. Bohm, P. Tombesi, A. Guerreiro, V. Vedral, A. Zeilinger, and M. Aspelmeyer, "Optomechanical entanglement between a movable mirror and a cavity field," *Phys. Rev. Lett.* **98**, 030405 (2007).
- [48] A. Wallucks, I. Marinkovi, B. Hensen, R. Stockill, and S. Groblacher, "A quantum memory at telecom wavelengths," *Nat. Phys.* **16**, 772 (2020).
- [49] D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer-Verlag, Berlin, 1994).
- [50] X. Wang, L.-M. Kuang, and J.-Q. Liao, "Parametrically driven squeezed states in cavity optomechanics," *Phys. Rev. A* **106**, 023502 (2022).
- [51] I. Wilson-Rae, N. Nooshi, W. Zwerger, and T. J. Kippenberg, "Theory of ground state cooling of a mechanical oscillator using dynamical backaction," *Phys. Rev. Lett.* **99**, 093901 (2007).
- [52] E. E. Wollman, C. U. Lei, A. J. Weinstein, J. Suh, A. Kronwald, F. Marquardt, A. A. Clerk, and K. C. Schwab, "Quantum squeezing of motion in a mechanical resonator," *Science* **349**, 952 (2015).
- [53] L.-A. Wu, H. J. Kimble, J. L. Hall, and H. Wu, "Generation of squeezed states by parametric down conversion," *Phys. Rev. Lett.* **57**, 2520 (1986).
- [54] Z. Wu, Z. Yi, W. Gu, L. Sun, and Z. Ficek, "Enhancement of optomechanical squeezing of light using optical coherent feedback," *Entropy* **24**, 1741 (2022).
- [55] R. Yanagimoto, E. Ng, L. G. Wright, T. Onodera, and H. Mabuchi, "Quantum theory of nonlinear optical circuits: From coupled modes to graph theory," *Optica* **8**, 1306 (2021).
- [56] J. Zhang, Y. Li, and K. Vahala, "Enhanced single-mode squeezing in optomechanical systems with two-photon driving," *Phys. Rev. Applied* **21**, 034045 (2024).
- [57] Y. Zhang, H. Wang, X. Li, J. Jing, C. Xie, and K. Peng, "Experimental generation of bright two-mode quadrature squeezed light from a narrow-band optical parametric amplifier," *Phys. Rev. A* **62**, 023813 (2000).
- [58] R. Simon, "Peres-Horodecki separability criterion for continuous variable systems," *Phys. Rev. Lett.* **84**, 2726 (2000).

- [59] G. Vidal and R. F. Werner, "Computable measure of entanglement," *Phys. Rev. A* 65, 032314 (2002).
- [60] G. Adesso, A. Serafini, and F. Illuminati, "Extremal entanglement and mixedness in continuous variable systems," *Phys. Rev. A* 70, 022318 (2004).
- [61] L. Henderson and V. Vedral, "Classical, quantum and total correlations," *J. Phys. A* 34, 6899 (2001).
- [62] H. Ollivier and W. H. Zurek, "Quantum discord: A measure of the quantumness of correlations," *Phys. Rev. Lett.* 88, 017901 (2001).
- [63] G. Adesso and A. Datta, "Quantum versus classical correlations in Gaussian states," *Phys. Rev. Lett.* 105, 030501 (2010).