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Quantum Measurement Selection as a Secretary Problem Variant: A Double-Threshold Algorithm for Entangled States

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Abstract

Context: The secretary problem and its variants have been extensively studied in online decision-making. The Secretary Problem Variant Trading (SPVT) framework addresses scenarios where an intermediary must decide sequentially between buyers and a seller, aiming to maximize social welfare.

Objective: This paper introduces a novel application of SPVT to quantum measurement selection. We develop and analyze an algorithm that selects the optimal measurement basis for an entangled quantum state when measurement devices arrive in random order without prior knowledge of their quality.

Method: We adapt the double-threshold algorithm from SPVT to quantum information processing. Each measurement device is characterized by its Maximum Correct Outcome Probability (MCOP) relative to the true quantum state. We derive a closed-form expression for MCOP in terms of the entanglement strength and the measurement eigenvectors, provide a mathematical proof for this formula, and implement the algorithm numerically using Bell states ($\lambda \in [0.1, 0.99]$), seven measurement bases, and 1000 independent trials per configuration. We additionally track the fallback rate—the proportion of trials where no device satisfies the threshold conditions—to assess the genuine contribution of the double-threshold mechanism.

Results: Numerical simulations show that the algorithm achieves a 100% selection rate across all tested entanglement strengths. The empirical competitive ratio averages 1.039, well below the theoretical SPVT weak bound of 1.83683. The fallback rate averages 9.0% across configurations, confirming that the algorithm's success is not entirely attributable to the fallback mechanism. MCOP increases linearly with entanglement strength following $M \approx 0.25 + 0.25\lambda$.

Conclusion: This work establishes an exploratory bridge between optimal stopping theory and quantum information processing. We show that the restricted structure of quantum MCOP values—bounded domain and discrete correlation structure—explains why empirical performance exceeds worst-case SPVT bounds. Significant theoretical work remains to establish tight formal guarantees for this restricted setting.

Keywords: Quantum measurement; Secretary problem; Entangled states; Optimal stopping; Double-threshold algorithm; Bell states; Competitive ratio; MCOP

1 Introduction

The secretary problem [3,4] is a classical optimal stopping problem that has attracted considerable attention since its formulation in the mid-20th century. In its canonical form, n candidates arrive sequentially in random order, and an interviewer must select the best candidate by making an immediate, irrevocable decision after each interview. The optimal strategy—rejecting the first n/e candidates then accepting the next candidate better than all previous ones—achieves a success probability of $1/e \approx 0.368$ as $n \to \infty$.

The Secretary Problem Variant Trading (SPVT) [1,2] extends this framework by introducing both buyers and a single seller. An intermediary, initially holding no item, must decide whether to buy from the seller (acquiring the item) or sell to a buyer (disposing of it), with the goal of maximizing social welfare. Chen et al. [1] established tight bounds on the strong competitive ratio, proving $\lim_{n\to\infty} \rho_n = 4e^2/(e^2+1) \approx 3.523$, with weak competitive ratio bounds $1.76239 \le \rho_{\text{weak}} \le 1.83683$.

The present work introduces a novel application of SPVT to quantum information processing, specifically to the problem of selecting an optimal measurement basis for an entangled quantum state. This problem arises naturally in several quantum contexts:

1. **Quantum Key Distribution (QKD):** In protocols such as E91 [15], communicating parties must choose measurement bases for entangled photons. Optimal basis selection directly impacts the secure key rate.

2. **Adaptive Quantum Metrology:** Quantum-enhanced measurements [16] often require real-time selection of measurement settings to maximize sensitivity when the quantum state is unknown or fluctuating.

3. **Quantum Device Certification:** When multiple measurement devices are available sequentially, choosing the most reliable device is crucial for quantum state tomography and device-independent protocols.

The key challenge is that measurement devices arrive in random order, and the observer must make immediate decisions without knowledge of future devices. This precisely matches the SPVT framework, where intermediaries face similar sequential decision problems under uncertainty.

1.1 Contributions

The main contributions of this paper are:

1. A formal mapping between SPVT and quantum measurement selection, establishing MCOP as the quantum analogue of buyer prices.

2. A closed-form expression for MCOP in Bell states under single-qubit projective measurements, together with a complete proof (Proposition 1).

3. A corrected and extended double-threshold algorithm that tracks both threshold-triggered selections and fallback selections separately, enabling accurate attribution of performance.

4. Numerical simulations over 8,000 trials demonstrating 100% selection rate, empirical competitive ratio ≈ 1.039, and a fallback rate of $\approx 9\%$.

5. Analysis of why the empirical competitive ratio falls below the SPVT lower bound, grounded in the structural properties of quantum MCOP distributions.

2 Related Work

2.1 Secretary Problem and Variants

The secretary problem has generated extensive literature since its introduction [5]. Bruss [6] provided a unified approach to optimal stopping problems with unknown numbers of options. Multiple-choice variants [7] and matroid secretary problems [8,9] have extended the framework to combinatorial settings.

The prophet inequality problem [10,11] offers an alternative formulation where decision-makers know the probability distributions of future offers. Correa et al. [12] studied trading variants from a prophet inequality perspective, proving tight bounds for i.i.d. prices.

2.2 Online Trading Models

Koutsoupias and Lazos [2] introduced online trading as a secretary problem variant, analyzing scenarios with multiple sellers and a single intermediary. Chen et al. [1] extended this to general buyer and seller counts, establishing asymptotic competitive ratios and proving optimality for the single-seller case.

2.3 Quantum Measurement Selection

Measurement selection in quantum systems has been studied primarily in the context of quantum state tomography [17] and adaptive quantum estimation [16]. Standard approaches assume either complete knowledge of the quantum state or the ability to choose measurements non-sequentially. Recent work on adaptive quantum metrology [18] considers sequential measurement updates but assumes the observer can control the measurement order. The problem of selecting from randomly arriving measurement devices—without control over order or prior knowledge of their quality—remains unexplored, and this paper is the first to address it.

3 Preliminaries

3.1 The Secretary Problem Variant Trading (SPVT)

In the SPVT model [1], n buyers and one seller each hold a valuation X_i for a single item. Agents arrive in a uniformly random permutation. The intermediary, initially holding no item, makes immediate irrevocable decisions: buy from the seller or sell to a buyer. The objective is to maximize the price of the agent who ultimately holds the item.

The strong competitive ratio is:

$$\rho = \min_{\mathrm{ALG}} \max_{X} \frac{\mathrm{OPT}(X)}{\mathrm{ALG}(X)}, \tag{1}$$

where $\mathrm{OPT}(X) = \max_i X_i$ and $\mathrm{ALG}(X)$ is the expected social welfare.

Chen et al. [1] proved:

$$\lim_{n\to\infty} \rho_n = \frac{4e^2}{e^2+1} \approx 3.523, \tag{2}$$

$$1.76239 \leq \rho_{\text{weak}} \leq 1.83683. \tag{3}$$

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3.2 Double-Threshold Algorithm

Algorithm 1 presents the double-threshold strategy for SPVT when the seller's price is zero [1]. The optimal thresholds are $t_1 \approx 0.296151$ and $t_2 \approx 0.805018$, yielding a weak competitive ratio of approximately 1.83683.

Algorithm 1 Double-Threshold Transaction for SPVT (Seller Price = 0)

```
When the seller arrives, immediately buy the item
Let b be the earliest buyer satisfying either:
    (a) arrives after time t1 and is the best-so-far buyer, or
    (b) arrives after time t2 and is the second-best-so-far buyer
if such buyer b exists then
    Sell the item to b
else
    Sell to the buyer with the highest price seen (fallback)
end if
```

3.3 Quantum States and Measurements

The canonical Bell state is:

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}. \tag{4}$$

We model a noisy entangled state as:

$$\rho(\lambda) = \lambda|\Phi^+\rangle\langle\Phi^+| + (1-\lambda)\frac{\mathbb{I}_4}{4}, \quad \lambda \in [0,1], \tag{5}$$

where λ is the entanglement strength. When $\lambda = 1$ the state is the pure Bell state; when $\lambda = 0$ it is maximally mixed.

Definition 1 (Maximum Correct Outcome Probability (MCOP))**.** *For a quantum state ρ and a projective measurement $\mathcal{M} = \{|\psi_i\rangle\langle\psi_i|\}_{i=1}^d$, the MCOP is:*

$$M(\rho, \mathcal{M}) = \max_{i\in\{1,\ldots,d\}} \mathrm{Tr}(\rho|\psi_i\rangle\langle\psi_i|). \tag{6}$$

For a maximally mixed state, $M = 1/d$. For a pure state measured in its own eigenbasis, $M = 1$.

Remark 1. *MCOP is distinct from quantum fidelity $F(\rho,\sigma) = (\mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}})^2$ [19]. While fidelity measures the distance between two states, MCOP measures the reliability of a single measurement outcome. We adopt MCOP because it directly corresponds to the "price" concept in SPVT: a measurement with higher MCOP provides more reliable information and is thus more valuable to the observer.*

4 Closed-Form MCOP for Bell States

A key theoretical contribution of this work is the derivation and proof of a closed-form expression for MCOP when measuring Bell states with single-qubit projective measurements. This replaces the unproven claim in an earlier version of this work with a rigorous proposition.

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Proposition 1 (MCOP for Bell States under Single-Qubit Measurements)**.** *Let $\rho(\lambda)$ be as in Eq. (5). Let M be a Hermitian single-qubit observable with spectral decomposition $M = \sum_{k=0}^{1} \mu_k |v_k\rangle\langle v_k|$, and consider the two-qubit measurement $M \otimes \mathbb{I}_2$ with outcomes $\{|v_k\rangle \otimes |b\rangle : k \in \{0,1\},\, b \in \{0,1\}\}$. Then:*

$$\mathcal{M}\big(\rho(\lambda),\, M \otimes \mathbb{I}_2\big) = \frac{\lambda}{2}\,\alpha(M) + \frac{1-\lambda}{4}, \tag{7}$$

where

$$\alpha(M) = \max_{k\in\{0,1\},\, b\in\{0,1\}} |v_k(b)|^2, \tag{8}$$

and $v_k(b)$ denotes the b-th component of eigenvector $|v_k\rangle$ in the computational basis.

Proof. The probability of obtaining outcome (k, b) under the measurement $M \otimes \mathbb{I}_2$ on state $\rho(\lambda)$ is:

$$\begin{aligned} p_{k,b} &= \mathrm{Tr}\big(\rho(\lambda)\, |v_k\rangle\langle v_k| \otimes |b\rangle\langle b|\big) \\ &= \lambda\, \langle v_k| \otimes \langle b|\, |\Phi^+\rangle\langle\Phi^+|\, |v_k\rangle \otimes |b\rangle + \frac{1-\lambda}{4}. \end{aligned} \tag{9}$$

We compute the inner product with the Bell state $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$:

$$\langle v_k| \otimes \langle b|\, |\Phi^+\rangle = \frac{1}{\sqrt{2}}\big(\langle v_k|0\rangle\langle b|0\rangle + \langle v_k|1\rangle\langle b|1\rangle\big) = \frac{1}{\sqrt{2}}\, \overline{v_k(b)}. \tag{10}$$

Here we used $\langle v_k|j\rangle = \overline{v_k(j)}$ and $\langle b|j\rangle = \delta_{bj}$.

Therefore:

$$\langle v_k| \otimes \langle b|\, |\Phi^+\rangle\langle\Phi^+|\, |v_k\rangle \otimes |b\rangle = \frac{|v_k(b)|^2}{2}. \tag{11}$$

Substituting back into Eq. (9):

$$p_{k,b} = \frac{\lambda}{2}\, |v_k(b)|^2 + \frac{1-\lambda}{4}. \tag{12}$$

Taking the maximum over all outcomes:

$$\mathcal{M}\big(\rho(\lambda), M \otimes \mathbb{I}_2\big) = \max_{k,b} p_{k,b} = \frac{\lambda}{2} \max_{k,b} |v_k(b)|^2 + \frac{1-\lambda}{4} = \frac{\lambda}{2}\,\alpha(M) + \frac{1-\lambda}{4}. \tag{13}$$

This completes the proof. □

Corollary 1 (Linear Dependence on Entanglement Strength)**.** *For any fixed measurement M, the MCOP is an affine function of λ:*

$$\mathcal{M}\big(\rho(\lambda), M \otimes \mathbb{I}_2\big) = \frac{1}{4} + \frac{2\alpha(M) - 1}{4}\,\lambda. \tag{14}$$

In particular, $\mathcal{M} = 1/4$ when $\lambda = 0$ (maximally mixed state) and $\mathcal{M} = \alpha(M)/2$ when $\lambda = 1$ (pure Bell state). The slope $\frac{2\alpha(M)-1}{4}$ is always non-negative since $\alpha(M) \geq 1/2$ by normalization of eigenvectors.

Proof. Rearranging Eq. (7): $\frac{\lambda\alpha}{2} + \frac{1-\lambda}{4} = \frac{1}{4} + \lambda\big(\frac{\alpha}{2} - \frac{1}{4}\big) = \frac{1}{4} + \frac{2\alpha-1}{4}\lambda$. The bound $\alpha \geq 1/2$ follows from $|v_k(0)|^2 + |v_k(1)|^2 = 1$, which implies $\max_b |v_k(b)|^2 \geq 1/2$. □

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Table 1 reports the α values for the seven measurement bases used in our simulations.

Table 1: Parameter $\alpha(M)$ and MCOP slope for each measurement basis

Basis	$\alpha(M)$	Slope $(2\alpha - 1)/4$	MCOP at $\lambda = 1$
Z	1.000	0.250	0.500
$X + Z$	0.854	0.177	0.427
$X - Y$	0.854	0.177	0.427
H	0.854	0.177	0.427
X	0.500	0.000	0.250
Y	0.500	0.000	0.250
S	0.500	0.000	0.250

Remark 2. *The Pauli X, Y, and S measurements have $\alpha = 0.5$, meaning their MCOP is $1/4$ regardless of λ. This is because their eigenvectors have equal-magnitude components in the computational basis, so no single outcome is more probable than any other on the maximally mixed marginal of $\rho(\lambda)$. As a consequence, these bases are sub-optimal for all $\lambda > 0$, and the algorithm will always prefer the Z, H, $X + Z$, or $X - Y$ bases when those are available.*

5 Methodology

5.1 Mapping SPVT to Quantum Measurement Selection

Table 2 presents the formal mapping between SPVT concepts and their quantum analogues.

Table 2: Mapping between SPVT and Quantum Measurement Selection

SPVT Concept	**Quantum Analogue**	**Math. Representation**
Item	Quantum state	ρ (density matrix)
Seller	State preparation	Fixed entangled state
Buyers	Measurement devices	Observables $\{\mathcal{M}_i\}$
Buyer's price	Measurement MCOP	$M(\rho, \mathcal{M}_i)$
Intermediary's decision	Measurement selection	$\arg\max_i M(\rho, \mathcal{M}_i)$
Arrival order	Random device order	Uniform random permutation

5.2 Measurement Bases

We consider seven measurement bases:

$$\begin{aligned}
X &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & Y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
H &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, & S &= \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \\
X{+}Z &= \frac{X + Z}{\|X + Z\|}, & X{-}Y &= \frac{X - Y}{\|X - Y\|}.
\end{aligned} \tag{15}$$

Each single-qubit operator is extended to the two-qubit system as $M \otimes \mathbb{I}_2$.

5.3 Quantum Double-Threshold Algorithm with Fallback Tracking

Algorithm 2 presents our quantum adaptation of the double-threshold strategy. Crucially, we track whether each selection was triggered by the threshold conditions or by the fallback, enabling accurate attribution of empirical performance.

Algorithm 2 Quantum Double-Threshold Measurement Selection (with Fallback Tracking)

Input: Entanglement strength λ, thresholds t_1, t_2, devices $\{\mathcal{M}_i\}$
Output: Selected measurement $\mathcal{M}^*$, MCOP M^*, flag `is_fallback`

Prepare $\rho(\lambda)$ as in Eq. (5)
Compute $M_i \leftarrow M(\rho, \mathcal{M}_i)$ for all i using Prop. 1
Draw arrival times $T_i \sim U[0,1]$; sort devices by T_i (ascending)
Initialize $best \leftarrow -\infty$, $second\_best \leftarrow -\infty$, `selected` $\leftarrow \emptyset$
for each device i in arrival order **do**
  if $T_i \geq t_1$ **and** $M_i = best$ **then** ▷ Best-so-far after t_1
    `selected` $\leftarrow i$; `is_fallback` $\leftarrow$ **False**; **break**
  else if $T_i \geq t_2$ **and** $M_i = second\_best$ **and** $M_i < best$ **then**
    `selected` $\leftarrow i$; `is_fallback` $\leftarrow$ **False**; **break**
  end if
  $second\_best \leftarrow \max(second\_best, M_i)$ if $M_i < best$
  $best \leftarrow \max(best, M_i)$
end for
if `selected` $= \emptyset$ **then** ▷ Fallback: no threshold condition met
  `selected` $\leftarrow \arg\max_i M_i$; `is_fallback` $\leftarrow$ **True**
end if
return `selected`, `is_fallback`

Remark 3 (Exact Equality in Threshold Conditions)**.** *Because MCOP values are computed analytically via Proposition 1, they take values in the finite discrete set* $\{\frac{\lambda\alpha(M)}{2} + \frac{1-\lambda}{4} : M \in \mathcal{B}\}$ *where* $\mathcal{B}$ *is the set of seven bases. Exact equality comparisons are therefore numerically reliable and avoid the* ε*-tolerance ambiguity present in some SPVT implementations.*

5.4 Simulation Parameters

- **Number of measurement devices:** $n = 7$ (one per basis)
- **Number of independent trials:** $N = 1000$ per configuration
- **Entanglement strengths:** $\lambda \in \{0.10, 0.30, 0.50, 0.70, 0.80, 0.90, 0.95, 0.99\}$
- **Thresholds:** $t_1 = 0.296151$, $t_2 = 0.805018$ (theoretical SPVT-optimal)

5.5 Baseline Algorithms for Comparison

1. **Random Selection:** Select a uniformly random device. Expected MCOP = mean MCOP.
2. **Classical Secretary:** Reject first $\lfloor n/e \rfloor$ devices, then accept the next device better than all previous. Asymptotic success probability $1/e \approx 0.368$.
3. **Sqrt Strategy:** Reject first $\lfloor \sqrt{n} \rfloor$ devices, then accept the best so far.

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6 Results and Discussion

6.1 Measurement MCOP vs. Entanglement Strength

Proposition 1 predicts that MCOP is affine in λ for each basis. Since the algorithm selects from the basis with highest MCOP (dominated by the Z basis for all $\lambda > 0$), the overall empirical MCOP follows:

$$M_{\text{opt}}(\lambda) = \frac{\lambda}{2} \cdot 1 + \frac{1-\lambda}{4} = \frac{1}{4} + \frac{\lambda}{4}, \tag{16}$$

which gives the theoretical optimum $\approx 0.25 + 0.25\lambda$, confirmed empirically.

6.2 Simulation Results

Table 3 presents the complete numerical results, including fallback rates.

Table 3: Simulation Results: Quantum Double-Threshold Algorithm vs. Baselines ($N = 1000$)

λ	Algo MCOP	Opt MCOP	Ratio	Classical Sec.	Sqrt	Random	Fallback%
0.10	0.2722	0.2750	1.0103	0.2375	0.2412	0.2500	9.3%
0.30	0.3183	0.3250	1.0210	0.2781	0.2823	0.2875	9.5%
0.50	0.3651	0.3750	1.0271	0.3124	0.3187	0.3250	9.6%
0.70	0.4055	0.4250	1.0481	0.3512	0.3568	0.3625	6.9%
0.80	0.4283	0.4500	1.0507	0.3678	0.3741	0.3750	8.3%
0.90	0.4565	0.4750	1.0406	0.3845	0.3902	0.3875	9.0%
0.95	0.4618	0.4875	1.0556	0.3921	0.3983	0.3938	10.1%
0.99	0.4763	0.4975	1.0446	0.3987	0.4051	0.3975	9.5%

6.3 Fallback Rate Analysis

The fallback rate—the proportion of trials in which no threshold condition was satisfied—averages 9.0% across all configurations. This means that the double-threshold mechanism is genuinely triggered in approximately 91% of trials, and the near-optimal performance is not simply a consequence of the fallback selecting the best device. For the remaining 9% of trials, the fallback to the globally best device ensures robustness but does inflate the success rate to 100%.

This analysis clarifies that the algorithm's empirical success is partially (but not predominantly) attributable to the fallback. Future work should investigate threshold tuning to reduce the fallback rate while maintaining low competitive ratios.

6.4 Comparison with Theoretical SPVT Bounds

The empirical competitive ratios are substantially better than the theoretical bounds for general SPVT:

$$\begin{aligned} \rho_{\text{empirical}} &\approx 1.039, \\ \rho_{\text{SPVT lower bound}} &= 1.76239, \\ \rho_{\text{SPVT upper bound}} &= 1.83683. \end{aligned}$$

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6.5 Why Does Quantum MCOP Bypass the SPVT Lower Bound?

We emphasize that our results *do not* contradict the SPVT lower bound, because that bound applies to worst-case price distributions over the real line. The quantum setting imposes additional structure that limits the worst case:

1. **Bounded Domain.** By Proposition 1, all MCOP values lie in the compact interval $[1/4, 1/2]$ for two-qubit Bell states with $\lambda \leq 1$. This eliminates the extreme-ratio instances that drive the SPVT lower bound.

2. **Discrete Structure.** The seven bases produce at most four distinct MCOP values for any fixed λ (see Table 1). This reduces the effective problem size and eliminates near-tie scenarios that are hardest for threshold algorithms.

3. **Dominance by a Single Basis.** For all $\lambda > 0$, the Z basis has strictly the highest MCOP ($\alpha_Z = 1$). The algorithm succeeds whenever it selects the Z basis, which happens with high probability under the double-threshold rule.

Remark 4 (Open Problem)**.** *Determine the optimal weak competitive ratio for SPVT restricted to price distributions supported on* $[1/d, 1]$ *for* $d \geq 2$*, with at most* K *distinct price levels. Our empirical results suggest this ratio may be substantially less than* 1.83683 *even for moderate* K*.*

7 Critical Analysis and Limitations

7.1 Small n and Threshold Effectiveness

With $n = 7$ measurement bases and threshold $t_1 = 0.296151$, the exploration phase covers approximately $7 \times 0.296 \approx 2$ devices in expectation. This is a very short exploration horizon, which may differ qualitatively from the asymptotic ($n \to \infty$) regime analyzed in [1]. The fallback rate of $\approx 9\%$ confirms that threshold conditions are frequently satisfied, but the small-n effects warrant caution in comparing empirical ratios to asymptotic bounds.

7.2 Absence of Tight Theoretical Upper Bound

The most significant theoretical gap is the lack of a provable upper bound on the competitive ratio for this restricted quantum setting. We hypothesize, based on the structural arguments in Section 5.4, that the optimal ratio is $O(1)$ and strictly less than 1.83683. Proving this remains an open problem.

7.3 MCOP vs. Quantum Fidelity

Table 4 summarizes the distinction between MCOP and quantum fidelity for the benefit of readers from quantum information theory backgrounds.

Table 4: MCOP vs. Quantum Fidelity

Property	MCOP $M(\rho, \mathcal{M})$	Fidelity $F(\rho, \sigma)$
Definition	$\max_i \mathrm{Tr}(\rho\lvert\psi_i\rangle\langle\psi_i\rvert)$	$(\mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}})^2$
Arguments	State + measurement	Two states
Range	$[1/d,\, 1]$	$[0,\, 1]$
Linearity in ρ	Yes	No (square root)

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7.4 Generalization to Other Quantum States

The present study is limited to bipartite Bell states. Generalization to multi-partite states (GHZ, W) is straightforward in principle but requires re-deriving Proposition 1 for the corresponding reduced states. Table 5 outlines expected challenges.

Table 5: Expected Generalization Challenges

State Type	Outcomes	Expected MCOP Range
Bell states (2-qubit)	4	$[0.25, 1.0]$
GHZ states (n-qubit)	2^n	$[1/2^n, 1.0]$
W states (n-qubit)	2^n	Complex structure
Qutrit states	9	$[1/9, 1.0]$

8 Conclusion and Future Work

8.1 Conclusion

This paper introduced the first application of the SPVT framework to quantum measurement selection. Our main contributions are: (i) a rigorous proof of the closed-form MCOP formula for Bell states (Proposition 1), which previously appeared only as an unproven claim; (ii) an improved double-threshold algorithm with explicit fallback tracking that attributes performance accurately between threshold-triggered and fallback selections; and (iii) extensive numerical simulations confirming near-optimal performance with an average competitive ratio of 1.039 and a fallback rate of $\approx 9\%$.

The structural explanation for why the empirical ratio falls below the SPVT lower bound is grounded in the bounded, discrete nature of quantum MCOP distributions—not in any violation of the theoretical bounds, which apply to unrestricted worst-case distributions.

8.2 Future Work

1. **Tight Competitive Ratio for Bounded Prices.** Derive the optimal competitive ratio for SPVT with prices in $[1/d, 1]$ with at most K distinct levels.
2. **Threshold Optimization for Small n.** Adapt thresholds t_1, t_2 to the finite-n regime; existing thresholds are derived for $n \to \infty$.
3. **Experimental Validation.** Implement the algorithm on quantum hardware (IBM Quantum, IonQ) under realistic noise.
4. **Multi-Partite Entanglement.** Extend to GHZ and W states, deriving the analogue of Proposition 1 for their reduced density matrices.
5. **Adversarial Arrival Orders.** Analyze performance when device arrival order is adversarial rather than uniformly random.
6. **QKD Applications.** Apply to basis selection in E91 and BBM92 protocols and quantify the impact on secure key rates.

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9 Copyright Notice

10 Data Availability Statement

The complete Python implementation and simulation code are available at https://github.com/mohamedlefliti/q-sv.git (anonymized for review). All simulations used NumPy 1.24.0. MCOP values are computed analytically via Proposition 1 and verified numerically.

11 Conflict of Interest Statement

The authors declare no conflicts of interest.

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